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EIGENMATRIX REPRESENTATIONS OF RADIANCE DISTRIBUTIONS IN LAYERED NATURAL WATERS WITH WIND-ROUGHENED SURFACES

Rudolph W. Preisendorfer

Pacific Marine Environmental Laboratory
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PREFACE

This technical memorandum preserves an essentially complete manuscript which was found among the papers of Dr. Rudolph Preisendorfer after his untimely death. Only minimal editing of the manuscript has been done by Dr. Curtis Mobley of the Joint Institute for the Study of the Atmosphere and Ocean, University of Washington, Seattle, Washington. Dr. Mobley was supported in this work by the Oceanic Biology Program of the Office of Naval Research under contract number N00014-87-K-0525.

CONTENTS

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Rudolph W. Preisendorfer

ABSTRACT. This report develops analytic, closed-form solutions for *radiance* distributions in natural waters such as lakes and seas. The solutions are valid in layered water bodies for which each layer has inherent optical properties (absorption and scattering functions) which are independent of depth within that layer. The water body is assumed free of internal light sources. The effects of a wind-blown air-sea surface are included. This work extends to the radiance level certain results which were previously known to hold for irradiances. The eigenmatrix formalism developed here is convenient for numerical computation of radiance distributions, given the inherent optical properties of the water and the desired boundary conditions at the water surface and bottom (the direct problem). Moreover, the formalism suggests an algorithm for solving the inverse problem: the determination of the inherent optical properties from measurements of the radiance distribution within a water body.

1. INTRODUCTION

We develop here a method for solving the equation of transfer for radiance in a piecewise homogeneous, source-free, plane parallel water body with a wind-roughened air-water surface. The assumption of homogeneity within each layer of the medium allows a closed-form type of solution which views the radiance distribution at each depth within a layer as a linear combination of purely exponential elementary components which decrease or increase with depth. This mode of decomposition of the multi-directional light field has the same simple visualizability as the classic two-flow irradiance model of the light field from which the theory of radiative transfer in stratified media began (cf. Schuster, 1905). Moreover the new representation allows explicit analytic and algebraic formulas to be developed for such basic properties of the medium as the radiance reflectance and radiance transmittance of the component layers, the radiance distribution within each layer, and also the asymptotic radiance distribution evolving in an infinitely deep homogeneous lower layer of a medium. Numerical solutions are readily forthcoming from such algebraic and analytic representations, and the boundary conditions for a wind-ruffled air-water surface endow the model with an ability to handle realistic reflectance and transmittance activity at the surface.

The present study builds on an earlier work, Mobley and Preisendorfer (1988), which describes in detail the so-called Natural Hydrosol Model (NHM) for the computation of radiance distributions in natural waters with wind-blown surfaces. (The NHM does not require the assumption of piecewise homogeneity of the water body.) In the interests of brevity, we shall draw freely from the results of this previous work; familiarity with Mobley and Preisendorfer (1988) is therefore a prerequisite for the full understanding of the present work. The essential feature of the NHM is the quad-averaging of the radiance distribution over quadrilateral subsets of the unit sphere of directions, and the exact splitting of the azimuthal structure of the quad-averaged radiance distribution into spectral modes by harmonic analysis. This double decomposition of the radiance distribution has the effect of reducing the integrodifferential equation of transfer for radiance to a set of coupled ordinary differential equations, one to each azimuthal spectral mode. From this point on, the techniques of the linear interaction principle (cf. Preisendorfer, 1976, vol. IV) can be applied to the family of differential equations associated to each spectral mode.

In homogeneous media, the system matrices of each spectral family of differential equations have depth-independent entries; this allows an extremely useful algebraic decomposition of the system matrix into its eigenvalues and eigenvectors in each homogeneous layer of the medium. The eigenvalues of the system matrix turn out to be the desired exponential modes of decay, and the eigenvectors become the framework for the directional structure of the radiance distributions.

For those who wish to see the ground from which the present eigenmatrix procedure has sprung, we include in Appendix A the differential equations of the two-flow model of the irradiance field, the modern descendants of Schuster's (1905) equations. It is indicated how the classic two-flow model establishes an algebraic pattern that generalizes from the simple upward and downward decomposition of photon flows to the multitude of photon flows in the present context. For those who are coming upon the notions of the linear interaction principle and its associated invariant imbedding procedures for the first time, the exposition in Appendix A should perhaps be studied before going on to section 2, below.

Sections 2 through 5 merely restate various results which are rigorously developed in Mobley and Preisendorfer (1988). The goal of this review is the local interaction principles as expressed in equation (5.16), along with the associated boundary conditions (5.18), (5.19) and (5.22). The real work of this report then begins in §6. A high point (at least for the author) in the present exposition comes in sections 7 and 8, where the physical meanings of the eigenvalues of the system matrices become clear, and where the eigenvectors of the system matrices are shown to give rise to linearly independent pieces of the radiance distribution. The development continues until the main goals of this work are reached in §16 and §17.

Acknowledgments. Initial numerical explorations of the eigenvalue and eigenvector problems defined here were made by Curtis D. Mobley, who was partially supported by a TOGA (Tropical Ocean Global Atmosphere) Council contract, and who also was partially supported by the National Climate Program

Office through the Climate Research Group at the Scripps Institute of Oceanography. Ryan Whitney did the word processing and Joy Register helped with the figures.

2. EQUATION OF TRANSFER AND BOUNDARY CONDITIONS

We begin with the equation of transfer for unpolarized spectral radiance, $N(y;\underline{\xi})$ ($W\cdot m^{-2}\cdot str^{-1}\cdot nm^{-1}$), in a source-free, non-fluorescing medium at (dimensionless) optical depth y along direction $\underline{\xi}$:

$$-\mu \frac{d}{dy} N(y;\underline{\xi}) = -N(y;\underline{\xi}) + \omega(y) \int_{\Xi} N(y;\underline{\xi}') \, p(y;\underline{\xi}';\underline{\xi}) \, d\Omega(\underline{\xi}') \tag{2.1}$$

$$x \le y \le z \quad ; \quad \underline{\xi} \in \Xi \quad ; \quad \mu \equiv \cos\theta$$

Here Ξ is the unit sphere of directions and $\omega(y) \equiv s(y)/\alpha(y)$ is the *scattering-to-attenuation* ratio, or albedo of single scattering. $s(y)$ (in m^{-1}) is the *volume total scattering function*, and $\alpha(y)$ (in m^{-1}) is the *volume attenuation function*. Hence the *volume absorption function* $a(y)$ (in m^{-1}) is given by $a(y) \equiv \alpha(y)-s(y)$. Moreover, $p(y;\underline{\xi}';\underline{\xi})$ (in str^{-1}) is the phase function and is related to the *volume scattering function* $\sigma(y;\underline{\xi}';\underline{\xi})$ by

$$\sigma(y;\underline{\xi}';\underline{\xi}) = s(y)p(y;\underline{\xi}';\underline{\xi}) \qquad (m^{-1}\ str^{-1}). \tag{2.2}$$

The phase function has the property

$$\int_{\Xi} p(y;\underline{\xi}';\underline{\xi}) \, d\Omega(\underline{\xi}) = 1 \tag{2.3}$$

for all $\underline{\xi}'$ in the unit sphere Ξ. Hence $p(y;\underline{\xi}';\underline{\xi})$ may be viewed as the probability that a photon incident along $\underline{\xi}'$ at depth y is scattered into a unit solid angle about $\underline{\xi}$, on the condition that the photon is scattered. (For a probabilistic interpretation of radiative transfer theory, see Preisendorfer, 1965, Ch. XIII.)

Figure 1 shows the geometric arrangement of the body X[x,z], x < z, of the natural hydrosol, its upper boundary X[a,x], and its lower boundary X[z,b]. In the present discussion X[a,x] is the infinitesimally thin average plane of the random air-water surface. We imagine level a to be just above and level x to be just below the surface. X[z,b] can be either an infinitesimally thin, opaque, matte reflecting bottom at a finite optical depth z below x, or an optically infinitely deep, homogeneous medium below depth z, with b = ∞. Both cases will be considered below.

The boundary conditions for (2.1) at the random upper air-water surface are

$$N(a;\underline{\xi}) = \int_{\Xi} N(x;\underline{\xi}') \, t(x,a;\underline{\xi}';\underline{\xi}) \, d\Omega(\underline{\xi}') + \int_{\Xi} N(a;\underline{\xi}') \, r(a,x;\underline{\xi}';\underline{\xi}) \, d\Omega(\underline{\xi}') \quad , \quad \xi \in \Xi_+ \tag{2.4}$$

$$N(x;\underline{\xi}) = \int_{\Xi_-} N(a;\underline{\xi}') \, t(a,x;\underline{\xi}';\underline{\xi}) \, d\Omega(\underline{\xi}') + \int_{\Xi_+} N(x;\underline{\xi}') \, r(x,a;\underline{\xi}';\underline{\xi}) \, d\Omega(\xi') \quad , \quad \xi \in \Xi_- \tag{2.5}$$

Here Ξ_+ and Ξ_- are the upper and lower hemispheres of Ξ, respectively.

The r and t functions describe how radiance is on average reflected and transmitted by the boundary surface. These are determined by a Monte Carlo method as developed in Preisendorfer and Mobley (1985, 1986) and applied to the Natural Hydrosol Model in Mobley and Preisendorfer (1988).

The boundary condition for (2.1) at the bottom level z is given by

$$N(z;\underline{\xi}) = \int_{\Xi_-} N(z;\underline{\xi}') \, r(z,b;\underline{\xi}';\underline{\xi}) \, d\Omega(\underline{\xi}') \quad , \quad \underline{\xi} \in \Xi_+ \tag{2.6}$$

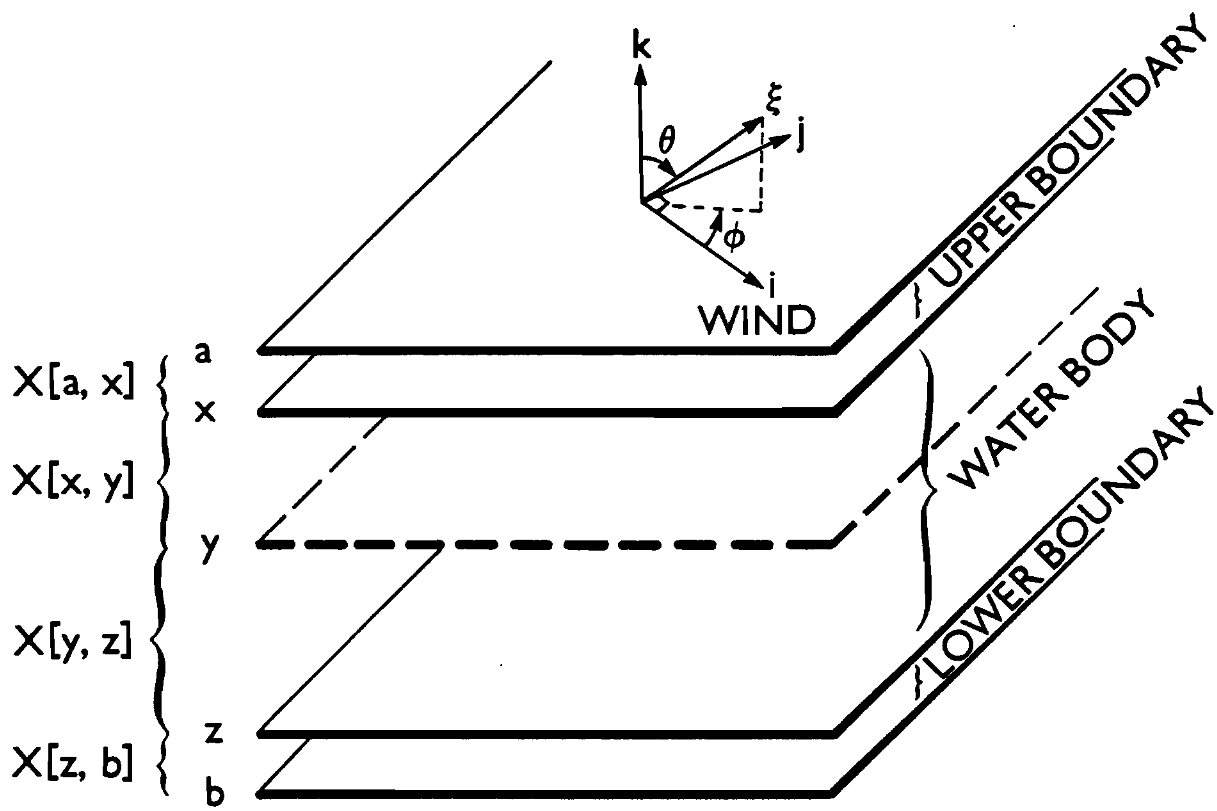

Figure 1.--The geometric setting of the National Hydrosol Model in a plane parallel medium with a wind-oriented coordinate system. The $\underline{i}$ vector is in the downwind direction; the $\underline{i}$-$\underline{j}$-$\underline{k}$ vectors form a right-handed coordinate system. A direction $\underline{\xi}$ is specified by the polar angle θ, $0 \leq \theta \leq \pi$, and the azimuthal angle ϕ, $0 \leq \phi < 2\pi$.

This is simpler than (2.4) or (2.5) because there are no upward or downward radiances postulated at level b, when X[z,b] is either of the two cases defined above.

3. QUAD-AVERAGED EQUATION OF TRANSFER AND BOUNDARY CONDITIONS

The unit sphere Ξ is shown in Fig. 2 partitioned into a set of quads which includes as special cases a pair of polar caps. There are m quad zones above and m quad zones below the equator, with each cap counting as a special zone. Each hemisphere is divided into 2n azimuthal sectors. In Fig. 2, m = 5 and 2n = 20.

A non-polar or regular quad Q_{uv} is indexed by a pair of integers u,v where u = 1,...,m-1 is the *zonal index* and v = 1,...,2n is the *azimuthal index*. Regular quads Q_{uv} have equal angular widths $\Delta\phi_v \equiv \Delta\phi \equiv \pi/n$ and arbitrary heights $\Delta\mu_u$. Quad Q_{uv} is centered at azimuth angle $\phi_v = (v-1)\Delta\phi$ and subtends a solid angle $\Omega_{uv} = \Delta\phi\Delta\mu_u$. Polar caps are denoted by "Q_m" and subtend solid angles of size $\Omega_m = 2\pi\Delta\mu_m$. It will be clear from the special notation developed below which hemisphere (Ξ_+ or Ξ_-) a cap or quad is in.

The *quad-averaged radiance* over quad Q_{uv} is defined by writing

$$N(y;u,v) \equiv \Omega_{uv}^{-1} \int_{Q_{uv}} N(y;\underline{\xi})\, d\Omega(\underline{\xi}) \tag{3.1}$$

$$= \Omega_{uv}^{-1} \int\int_{Q_{uv}} N(y;\mu,\phi)\, d\mu d\phi$$

Here we used an alternate description of $\underline{\xi}$ via its zenith ($\mu = \cos\theta$) and azimuth (ϕ) coordinates. When Q_{uv} is in Ξ_+ or in Ξ_- we will write the associated quad-averaged radiance as "$N^+(y;u,v)$" or "$N^-(y;u,v)$", respectively. In each of these cases, u runs from 1 to m^{-1} and v from 1 to 2n for regular quads. The quad-averaged radiance over a polar cap, for which u = m and v is undefined, is denoted by "$N^+(y;m,\cdot)$" or "$N^-(y;m,\cdot)$".

The *quad-averaged phase function* is defined by writing

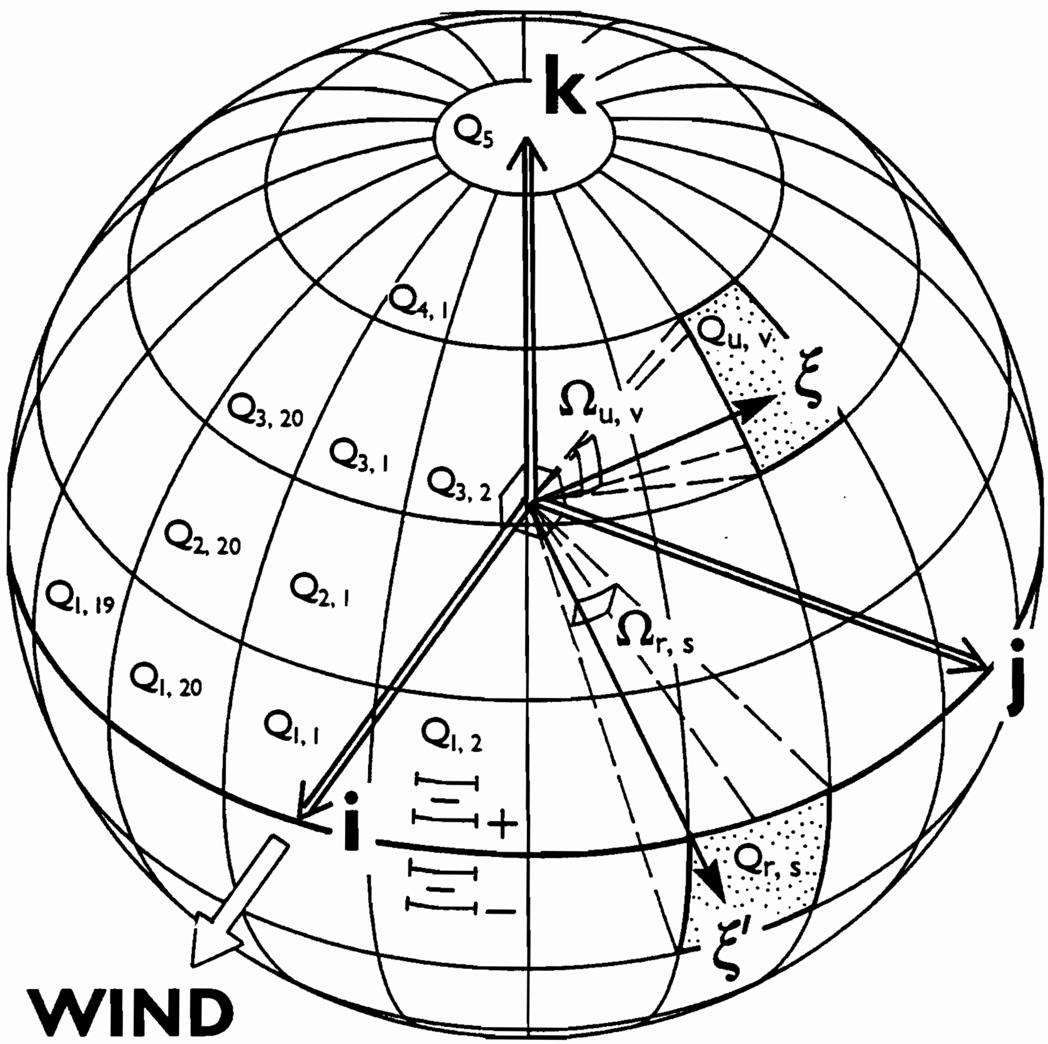

Figure 2.--An example partitioning of the unit sphere into quads and polar caps, for the case of m = 5 and n = 10. The coordinate system and quad indexing scheme is centered in the wind direction. Quad Q_{rs} is shown in the lower hemisphere of directions, Ξ_-, and Q_{uv} is shown in the upper hemisphere, Ξ_+.

$$p(y;r,s|u,v) \equiv \Omega^{-1}_{uv} \iint_{Q_{uv}} d\mu d\phi \iint_{Q_{rs}} d\mu' d\phi' \; p(y;\mu',\phi';\mu,\phi) \tag{3.2}$$

$x \le y \le z$; Q_{rs} and Q_{uv} in Ξ_+ or Ξ_-

Observe that there are three special cases involving polar caps; these are denoted by

$$\begin{aligned} &p(y;m,\cdot|u,v) \quad \text{(cap to quad)} \\ &p(y;r,s|m,\cdot) \quad \text{(quad to cap)} \\ &p(y;m,\cdot|m,\cdot) \quad \text{(cap to cap)} \end{aligned} \tag{3.3}$$

Applying the quad-averaging operation to (2.1) we find the desired *quad-averaged equation of transfer*:

$$-\mu_u \frac{d}{dy} N(y;u,v) = -N(y;u,v) + \omega(y) \sum_r \sum_s N(y;r,s) \; p(y;r,s;u,v) \tag{3.4}$$

$x \le y \le z$

Q_{uv} in Ξ_+ or Ξ_-

Here $\mu_u = \frac{1}{2}(\mu_1 + \mu_2)$, where μ_1 and μ_2 are the lower and upper cosines of the Q_{uv} boundaries. μ_u can be positive or negative. The summation in (3.4) is over all quads. Equation (3.4) is a set of $(m-1)\cdot 2n + (m-1)\cdot 2n + 2 = 4(m-1)n+2$ coupled ordinary differential equations with the same number of unknowns.

The quad-averaged forms of the surface boundary conditions (2.4) and (2.5) are

$$N(a;u,v) = \sum_r \sum_s N(x;r,s)\ t(x,a;r,s|u,v) + \sum_r \sum_s N(a;r,s)\ r(a,x;r,s|u,v) \quad (3.5)$$
$$Q_{uv} \text{ in } \Xi_+$$

$$N(x;u,v) = \sum_r \sum_s N(a;r,s)\ t(a,x;r,s|u,v) + \sum_r \sum_s N(x;r,s)\ r(x,a;r,s|u,v) \quad (3.6)$$
$$Q_{uv} \text{ in } \Xi_-$$

where the four surface transfer functions t(a,x), r(x,a), t(x,a), and r(a,x) are all defined following the general pattern

$$f(r,s|u,v) \equiv \Omega_{uv}^{-1} \int\int_{Q_{uv}} d\mu d\phi \int\int_{Q_{rs}} d\mu' d\phi'\ f(\mu',\phi';\mu,\phi) \quad (3.7)$$
$$Q_{rs},\ Q_{uv} \text{ in } \Xi$$

where "$f(\mu',\phi';\mu,\phi)$" denotes any of the four transfer functions in (2.4) or (2.5). The bottom boundary condition in quad-averaged form is

$$N(z;u,v) = \sum_r \sum_s N(z;r,s)\ r(z,b;r,s|u,v) \quad (3.8)$$
$$Q_{uv} \text{ in } \Xi_+$$

4. QUAD-AVERAGED LOCAL INTERACTION PRINCIPLES

The quad-averaged equation of transfer (3.4) may be split into two statements, one for the upward radiances $N^+(y;u,v)$ and one for the downward radiances $N^-(y;u,v)$ at each level y, $x \le y \le z$. The isotropy of the volume scattering functions $\sigma(y;\underline{\xi}';\underline{\xi})$, namely the property that its values depend only on $\underline{\xi}'\cdot\underline{\xi}$ and not $\underline{\xi}'$ and $\underline{\xi}$ separately, considerably simplifies the structure of the transfer equation. Thus we can write

$$\text{"}p^+(y;r,s|u,v)\text{"} \quad \text{for} \quad p(y;r,s|u,v) \quad \text{if} \quad \begin{cases} Q_{rs} \text{ in } \Xi_+ \text{ and } Q_{uv} \text{ in } \Xi_+ \\ \text{or} \\ Q_{rs} \text{ in } \Xi_- \text{ and } Q_{uv} \text{ in } \Xi_- \end{cases} \tag{4.1}$$

$$\text{"}p^-(y;r,s|u,v)\text{"} \quad \text{for} \quad p(y;r,s|u,v) \quad \text{if} \quad \begin{cases} Q_{rs} \text{ in } \Xi_+ \text{ and } Q_{uv} \text{ in } \Xi_- \\ \text{or} \\ Q_{rs} \text{ in } \Xi_- \text{ and } Q_{uv} \text{ in } \Xi_+ \end{cases}$$

Hence p^+ and p^- respectively act like local transmittance and reflectance functions in the body of the medium, relative to the horizontal plane of the equator of Ξ. We then find from (3.4) that

$$\begin{aligned} \mp\mu_u \frac{d}{dy} N^{\pm}(y;u,v) = -N^{\pm}(y;u,v) &+ \omega(y) \sum_r \sum_s N^+(y;r,s)\, p^{\pm}(y;r,s|u,v) \\ &+ \omega(y) \sum_r \sum_s N^-(y;r,s)\, p^{\mp}(y;r,s|u,v) \end{aligned} \tag{4.2}$$

$u = 1,\ldots,m;\ v = 1,\ldots,2n.$

This is a coupled pair of differential equation systems. The *upward system* is obtained by taking all upper signs together. This system describes the evolution with optical depth y of the upward radiances $N^+(y;u,v)$. The *downward system* describes $N^-(y;u,v)$. The complete system (4.2) constitutes the *local interaction principles* or the *local forms* of the *principles of invariance*. See Preisendorfer (1965, p. 103), H.O., Vol. III, p. 4, and Vol. II, p. 295. The boundary conditions (3.5), (3.6), and (3.8) hold also for (4.2).

5. SPECTRAL FORM OF THE QUAD-AVERAGED LOCAL INTERACTION PRINCIPLES

We next split the equation set (4.2) into smaller groupings of dependent variables by means of Fourier polynomial analysis. This entails no loss of information of the radiance field but considerably facilitates the numerical solution of the set (4.2).

For fixed y and u, $N^{\pm}(y;u,v)$ is a function defined on the finite set consisting of an even number of integers $v = 1,\ldots,2n$, corresponding to azimuth angles $\phi_v = (v-1)\Delta\phi$, as defined, above. We may then represent this v-dependence of $N^{\pm}(y;u,v)$ by

$$N^{\pm}(y;u,v) = \sum_{\ell=0}^{n} [A_1^{\pm}(y;u,\ell) \cos(\ell\phi_v) + A_2^{\pm}(y;u,\ell) \sin(\ell\phi_v)] \tag{5.1}$$

where

$$A_1^{\pm}(y;u;\ell) \equiv \varepsilon_\ell^{-1} \sum_{v=1}^{2n} N^{\pm}(y;u,v) \cos(\ell\phi_v) \qquad \ell = 0,\ldots,n \tag{5.2}$$

and

$$A_2^{\pm}(y;u;\ell) \equiv \gamma_\ell^{-1} \sum_{v=1}^{2n} N^{\pm}(y;u,v) \sin(\ell\phi_v) \qquad \ell = 1,\ldots,n-1 \tag{5.3}$$

These $A_p^{\pm}(y;u,\ell)$, $p = 1$ or 2, are the *radiance amplitudes* for (harmonic) mode ℓ. The factors ε_ℓ and γ_ℓ are given by

$$\varepsilon_\ell \equiv \begin{cases} 2n & \text{if } \ell=0 \text{ or } \ell=n \\ n & \text{if } \ell=1,\ldots,n-1 \end{cases} \tag{5.4}$$

$$\gamma_\ell \equiv \begin{cases} 0 & \text{if } \ell=0 \text{ or } \ell=n \\ n & \text{if } \ell=1,\ldots,n-1 \end{cases} \tag{5.5}$$

Observe that $\gamma_\ell = \varepsilon_\ell = n$ over the range $\ell = 1,\ldots,n-1$.

Since $\phi_v = (v-1)\pi/n$, we see that $\sin(\ell\phi_v) = 0$ for $\ell = 0$ and $\ell = n$. Hence we will define $A_2^\pm(y;u;0) \equiv 0$ and $A_2^\pm(y;u;n) \equiv 0$ for $u = 1,\ldots,m$. Also note that

$$\begin{aligned} A_1^\pm(y;m;0) &\equiv N^\pm(y;m,\cdot) \\ A_1^\pm(y;m;\ell) &\equiv 0 \quad , \quad \ell=1,\ldots,n \end{aligned} \tag{5.6}$$

and that

$$A_2^\pm(y;m;\ell) \equiv 0 \quad , \quad \ell=0,\ldots,n \tag{5.7}$$

Therefore all radiance amplitudes for the polar caps are zero, except for the cosine amplitude for the zero azimuthal index, $\ell = 0$. Holding y, r, s, and u fixed, we can represent the phase function, as a function of v, in Fourier polynomial form:

$$p^\pm(y;r,s|u,v) = \sum_{\ell=0}^{n} \hat{p}^\pm(y;r,u;\ell)\cos\ell(\phi_s-\phi_v) \tag{5.8}$$

$$x \le y \le z$$

$$r,u = 1,\ldots,m \quad ; \quad s,v = 1,\ldots,2n$$

where

$$\hat{p}^{\pm}(y;r,u;\ell) = [\epsilon_\ell \cos(\ell\phi_s)]^{-1} \sum_{v=1}^{2n} p^{\pm}(y;r,s|u,v) \cos\ell(\phi_s-\phi_v) \tag{5.9}$$
$$\ell = 0,\ldots,n$$

The representation (5.8) takes its form by virtue of the isotropy of the volume scattering function, noted above in sec. 4.

The Fourier representations of $N^{\pm}(y;u,v)$ and $p^{\pm}(y;r,s|u,v)$ in (5.1) and (5.8) are now substituted into (4.2). After rearrangements, and various definitions involving $\hat{p}^{\pm}(y;r,u;\ell)$ have been made, we arrive at three autonomous sets of equations over the range $x \le y \le z$. The first set is for the zero-mode cosine amplitudes of the radiance distribution:

$$\mp \frac{d}{dy} \underline{A}_1^{\pm}(y;0) = \underline{A}_1^{\pm}(y;0)\, \underline{\hat{\tau}}(y;0) + \underline{A}_1^{\mp}(y;0)\, \underline{\hat{\rho}}(y;0) \tag{5.10}$$

where

$$\underline{A}_1^{\pm}(y;0) \equiv [A_1^{\pm}(y;1;0),\ldots,A_1^{\pm}(y;m;0)] \tag{5.11}$$

The remaining n cosine amplitudes are governed by

$$\mp \frac{d}{dy} \underline{A}_1^{\pm}(y;\ell) = \underline{A}_1^{\pm}(y;\ell)\, \underline{\hat{\tau}}(y;\ell) + \underline{A}_1^{\mp}(y;\ell)\, \underline{\hat{\rho}}(y;\ell) \tag{5.12}$$

where

$$\underline{A}_1(y;\ell) \equiv [A_1^{\pm}(y;1;\ell),\ldots,A_1^{\pm}(y;m-1;\ell)] \tag{5.13}$$
$$\ell = 1,\ldots,n$$

and finally the n-1 non-zero sine amplitudes are generated by the set

$$\left\{\begin{array}{l} \mp \frac{d}{dy} \underline{A}_2^{\pm}(y;\ell) = \underline{A}_2^{\pm}(y;\ell)\ \underline{\hat{\tau}}(y;\ell) + \underline{A}_2^{\mp}(y;\ell)\ \underline{\hat{\rho}}(y;\ell) \qquad (5.14) \\ \text{where} \\ \underline{A}_2(y;\ell) \equiv [A_2^{\pm}(y;1;\ell),\ldots,A_2^{\pm}(y;m-1;\ell)] \qquad (5.15) \\ \ell = 1,\ldots,n-1 \end{array}\right.$$

The matrices $\underline{\hat{\rho}}(y;\ell)$ and $\underline{\hat{\tau}}(y;\ell)$ are either m×m or (m−1)×(m−1), as can be inferred in each case from the number of components of the $\underline{A}_p^{\pm}(y;\ell)$ vectors, p = 1,2. These matrices are fully defined in Mobley and Preisendorfer (1988). What should be noted here is that $\underline{\hat{\tau}}(y;\ell)$ is a *local transmittance matrix* in the sense that it propagates $A_p^{\pm}(y;\ell)$ into $A_p^{\pm}(y;\ell)$ over an infinitesimal increment Δy of optical depth. $\underline{\hat{\rho}}(y;\ell)$ acts as a *local reflectance matrix* in the sense that it respectively converts $A_p^{\mp}(y;\ell)$ into $A_p^{\pm}(y;\ell)$ over Δy. In this way the rising and descending streams of photons in an infinitesimal layer X[y,y+Δy] of a medium X[x,z] feedback to each other and generate the multiply scattered radiance field.

The preceding three autonomous sets of coupled differential equations all fall into the following general pattern

$$\boxed{\begin{array}{l} \mp \frac{d}{dy} \underline{A}_p^{\pm}(y;\ell) = \underline{A}_p^{\pm}(y;\ell)\ \underline{\hat{\tau}}(y;\ell) + \underline{A}_p^{\mp}(y;\ell)\ \underline{\hat{\rho}}(y;\ell) \\ x \le y \le z \\ p = 1,2\ ,\ \ell = 0,\ldots,n \\ \text{with} \\ \underline{A}_p^{\pm}(y;\ell) = [A_p^{\pm}(y;1;\ell),\ldots,A_p^{\pm}(y;q;\ell)] \\ q = m-1 \text{ or } m \end{array}}$$

(5.16)

(5.17)

Thus the amplitude vectors $\underline{A}^{\pm}(y;\ell)$ in (5.16) are 1×q and the matrices $\hat{\underline{\tau}}(y;\ell)$ and $\hat{\underline{\rho}}(y;\ell)$ are q×q, where q is either m−1 or m, as the case may be, i.e., depending on which of (5.10), (5.12), or (5.14) we are considering.

Once the three sets (5.10), (5.12), and (5.14) are solved we will have at each depth y exactly enough amplitudes to construct the radiances $N^{\pm}(y;u,v)$, $u = 1,\ldots,m$; $v = 1,\ldots,2n$ via (5.1). We will have for each flow (±), n+1 cosine amplitude vectors $\underline{A}_1^{\pm}(y;\ell)$, $\ell = 0,\ldots,n$ and n−1 sine amplitude vectors $\underline{A}_2^{\pm}(y;\ell)$, $\ell = 1,\ldots,n-1$, each with m−1 or m components, as needed.

The surface boundary conditions at X[a,x] that go with (5.16) are

$$\boxed{\begin{aligned} \underline{A}_p^{+}(a;\ell) &= \sum_{k=0}^{n} \underline{A}_p^{+}(x;k)\,\hat{\underline{t}}_p(x,a;k|\ell) + \sum_{k=0}^{n} \underline{A}_p^{-}(a;k)\,\hat{\underline{r}}_p(a,x;k|\ell) && (5.18)\\ \underline{A}_p^{-}(x;\ell) &= \sum_{k=0}^{n} \underline{A}_p^{-}(x;k)\,\hat{\underline{t}}_p(a,x;k|\ell) + \sum_{k=0}^{n} \underline{A}_p^{+}(x;k)\,\hat{\underline{r}}_p(x,a;k|\ell) && (5.19)\\ & p = 1 \text{ or } 2\ ,\quad \ell = 0,\ldots,n, \text{ and } k+\ell \text{ even.} \end{aligned}}$$

These are obtained by using the Fourier representations of each member of (3.5) and (3.6) and reducing to the indicated forms in (5.18) and (5.19). The azimuthal (ϕ-behavior) symmetries of the random wind-ruffled air water surface are those of an ellipse (cf. Mobley and Preisendorfer, 1988), which among other things require $\hat{\underline{t}}_p$ and $\hat{\underline{r}}_p$ to vanish when $k+\ell$ is odd. Hence the sums in (5.18) and (5.19) may be restricted to those values of k and ℓ for which $k+\ell$ is even.

The entries of the four transfer matrices $\hat{\underline{r}}_p$, $\hat{\underline{t}}_p$ in (5.18) and (5.19) are given in Mobley and Preisendorfer (1988). Observe that these are m×m matrices, some of which have zeros in their mth rows or mth columns (see Tables 1 and 2, Mobley and Preisendorfer, 1988). The essential point to note for later work (§12, below) is that the amplitude vectors $\underline{A}_p^{\pm}(x;k)$ and $\underline{A}_p^{\pm}(a;k)$ in (5.18) and (5.19) must be augmented to have m components, to be compatible with these surface transfer matrices.

The lower boundary surface is usually less complex than the random air-water surface. We therefore postulate a directional isotropy of the surface at level z in X[x,z] in analogy to the isotropy of the phase function in (5.8). Hence we shall represent the v-behavior of $r(z,b;r,s|u,v)$ as

$$r(z,b;r,s|u,v) = \sum_{\ell=0}^{n} \hat{r}(z,b;r,u|\ell) \cos\ell(\phi_s - \phi_v) \qquad (5.20)$$

$$r,u = 1,\ldots,m \quad ; \quad s,v = 1,\ldots,2n$$

where

$$\hat{r}(z,b;r,u|\ell) \equiv [\epsilon_\ell \cos(\ell\phi_s)]^{-1} \sum_{v=1}^{2n} r(z,b;r,s|u,v) \cos(\ell\phi_s) \qquad (5.21)$$

$$\ell = 0,\ldots,n$$

With these definitions, (3.8) converts to spectral form as

$$\boxed{\begin{array}{l} \underline{A}_p^{+}(z;\ell) = \underline{A}_p^{-}(z;\ell)\ \underline{\hat{r}}_p(z,b;\ell) \\ p = 1 \text{ or } 2 \text{ and } \ell = 0,\ldots,n \end{array}} \qquad (5.22)$$

where the entries of $\underline{\hat{r}}_p(z,b;\ell)$ are defined in Mobley and Preisendorfer (1988) for the two main cases of interest: (1) a matte bottom and (2) an imaginary surface above an infinitely deep homogeneous layer. Observe that the azimuthal isotropy of X[z,b] allows the reflected amplitudes $\underline{A}_p^{+}(z;\ell)$ in (5.22) to be uncoupled from all other modes $\underline{A}_p^{-}(z;k)$, $k \neq \ell$, incident on X[z,b]. The water surface X[a,x] is not azimuthally isotropic and so coupling takes place, as explicitly shown in (5.18) and (5.19).

6. THE FUNDAMENTAL MATRIX FOR RADIANCE AMPLITUDES IN HOMOGENEOUS LAYERS

We may now proceed to the main interest in this study, the solution of the equation set (5.16) when $\omega(y)$ and $p(y;\xi';\xi)$ in (3.4) and hence $\hat{\underline{\rho}}(y;\ell)$ and $\hat{\underline{\tau}}(y;\ell)$ are independent of optical depth y in an arbitrary layer X[x,z] of a natural hydrosol. The solution procedure we develop is independent of whether $p = 1$ or 2 or what mode index $\ell = 0,\ldots,n$ is of current interest. Accordingly, we can until further notice drop both "p" and "ℓ" from the notation in (5.16). (Both p and ℓ will have to be reinstated in §12, for example, when boundary conditions (5.18) and (5.19) are to be used, and also when the final Fourier synthesis of $N^{\pm}(y;u,v)$ in (5.1) is made).

A. *Basic Local Interaction Principles*

We will, in accordance with the preceding notational convention, now work with the following streamlined versions of (5.16) and (5.17):

$$\boxed{\mp \frac{d}{dy}\, \underline{A}^{+}(y) = \underline{A}^{\pm}(y)\, \hat{\underline{\tau}} + \underline{A}^{\mp}(y)\, \hat{\underline{\rho}}} \tag{6.1}$$

$$\underline{A}^{\pm}(y) = [A^{\pm}(y;1),\ldots,A^{\pm}(y;q)] \tag{6.2}$$

$$x \le y \le z$$

Thus $\hat{\underline{\rho}}$ and $\hat{\underline{\tau}}$ are $q\times q$ matrices with constant entries and $\underline{A}^{\pm}(y)$ at each depth y are $1\times q$ vectors. We may further simplify the system (6.1) and (6.2) by writing

$$\underline{A}(y) \equiv [\underline{A}^{+}(y),\ \underline{A}^{-}(y)] \qquad (1\times 2q) \tag{6.3}$$

and

$$\underline{K} \equiv \begin{bmatrix} -\hat{\underline{\tau}} & \hat{\underline{\rho}} \\ -\hat{\underline{\rho}} & \hat{\underline{\tau}} \end{bmatrix} \qquad (2q\times 2q) \tag{6.4}$$

so that (6.1) and (6.2) become the following version of the *local interaction principles*:

$$\boxed{\begin{array}{l} \dfrac{d}{dy}\,\underline{A}(y) = \underline{A}(y)\,\underline{K} \\ x \le y \le z \end{array}} \tag{6.5}$$

B. *The Constructive Definition of $M(x,y)$*

Equation (6.5) can be integrated at once either numerically or formally. We shall concentrate here on the former. Numerically, one would choose an initial $1\times 2q$ vector $\underline{A}(x)$ and then march (6.5) down from level x to any level y, $x \le y \le z$, in the given medium X[x,z]. There are many such initial vectors $\underline{A}(x)$ from which one may start. For example, one may have measured the radiances $N^{\pm}(x;u,v)$, $u = 1,\ldots,m$; $v = 1,\ldots,2n$ at level x just below the surface. From these radiances one can find $\underline{A}^{\pm}(x)$ as shown in (5.2) and (5.3); whence $\underline{A}(x)$. Then the amplitude $\underline{A}(y)$ is found by integrating (6.5), with $\underline{A}(x)$ as initial vector, down to any desired depth y, $x \le y \le z$.

There is one important set of initial vectors $\underline{A}(x)$, however, that leads to the *general solution* of (6.5). This is the set of 2q, $1\times 2q$ initial vectors stacked vertically to form the $2q\times 2q$ matrix:

$$\underline{M}(x,x) \equiv \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & & & & \vdots & \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix} = \underline{I}_{2q} \tag{6.6}$$

where $\underline{I}_{2q}$ is a 2q×2q identity matrix. Thus the initial 1×2q vectors form the 2q rows of the matrix $\underline{M}(x,x)$.

Let $\underline{m}_j(y)$ be the 1×2q vector solution of (6.5) for the initial 1×2q vector $\underline{m}_j(x) = [0,\ldots,1,\ldots,0]$ where all components are zero except that in place j, $1 \le j \le 2q$. Thus $\underline{m}_j(y)$ satisfies (6.5):

$$\frac{d}{dy}\,\underline{m}_j(y) = \underline{m}_j\underline{K} \quad , \quad j = 1,\ldots,2q$$

Write

$$\underline{M}(x,y) \equiv \begin{bmatrix} \underline{m}_1(y) \\ \vdots \\ \underline{m}_q(y) \\ \underline{m}_{q+1}(y) \\ \vdots \\ \underline{m}_{2q}(y) \end{bmatrix} \qquad (2q\times 2q) \tag{6.7}$$

Then clearly

$$\frac{d}{dy}\,\underline{M}(x,y) = \underline{M}(x,y)\,\underline{K} \tag{6.8}$$

and

$$\underline{M}(x,x) = \underline{I}_{2q} \tag{6.9}$$

The 2q×2q matrix $M(x,y)$ in (6.7) is the *fundamental matrix* of the system of differential equations (6.5). The importance of this system rests in the fact that if $\underline{A}(x)$ is any 1×2q vector, then the depth dependent 1×2q vector $\underline{A}(y) \equiv \underline{A}(x)\,\underline{M}(x,y)$ is the solution of (6.5) with initial value $\underline{A}(x)$. This follows formally on multiplying (6.8) from the left by $\underline{A}(x)$ and reducing the result to (6.5). The mathematical reason for this remarkable property is that the 2q vectors $\underline{m}_j(y)$ defined above are linearly independent for each y, $x \le y \le z$. Therefore any solution $\underline{m}(y)$ of (6.5) at some y is a vector in the space spanned by the $\underline{m}_j(y)$ (cf. Coddington and Levinson, 1955, pp. 68, 69). Thus we have the *mapping property* of $\underline{M}(x,y)$:

$$\underline{A}(y) = \underline{A}(x)\ \underline{M}(x,y) \qquad x \le y \le z \tag{6.10}$$

This mapping property also can be written down for the fundamental solution $\underline{M}(y,z)$ of (6.5) for which $M(y,y) = \underline{I}_{2q}$ and where $\underline{M}(y,z)$ is obtained by integrating (6.8) starting from level y. The associated mapping property is then $\underline{A}(y)\ M(y,z) = \underline{A}(z)$. Combining this with (6.10), noting that we have also $\underline{A}(z) = A(x)\ \underline{M}(x,z)$, and letting $\underline{A}(x)$ be arbitrary, we obtain the *group property* of $\underline{M}(x,z)$:

$$\underline{M}(x,z) = \underline{M}(x,y)\ \underline{M}(y,z) \qquad x \le y \le z. \tag{6.11}$$

Setting x = z in (6.11), the inverse of $M(x,y)$ is found to be

$$\underline{M}(y,x) = \underline{M}^{-1}(x,y) \qquad x \le y \le z \tag{6.12}$$

One may for example determine $\underline{M}(y,x)$ by integration, on starting at level y and integrating (6.5) upward from y to x. This would result in an associated mapping property for $\underline{M}(y,x)$ with initial amplitude $\underline{A}(y)$. Alternately, one may invert $\underline{M}(x,y)$ by algebraic or numerical means to find $M(y,x)$. Thus $\underline{M}^{-1}(x,y)$ has the physical interpretation of the mapping operator $\underline{M}(y,x)$, i.e., from level y to level $x < y$. From our observations above on the vector space aspects of $\underline{M}(x,y)$ it is clear that $\underline{M}^{-1}(x,y)$ exists in all natural hydrosols where $\hat{\underline{\rho}}(y;\ell)$ and $\hat{\underline{\tau}}(y;\ell)$ vary continuously with y. Since constant functions are continuous, this result holds also here.

C. *Exponential Representations of* $\underline{M}(x,y)$

The preceding definition of $\underline{M}(x,y)$ is the constructive definition, the one that will at once yield numerical values for the amplitude vectors $\underline{A}(y)$. It can be used even when $\hat{\underline{\rho}}$ and $\hat{\underline{\tau}}$ depend on y. There is, however, a formal definition of $\underline{M}(x,y)$ that is of great heuristic value; indeed it will lead us to the eigenmatrix representation of $\underline{A}(y)$, the central formula of the present study. This is the definition that starts from (6.8) and visualizes $\underline{M}(x,y)$ as given by the same kind of formula found in the theory of the scalar-valued exponential function. Thus let us write

$$\underline{M}(x,y) \equiv \exp[\underline{K}(y-x)] \tag{6.13}$$

and

$$\exp[\underline{K}(y-x)] \equiv \sum_{j=0}^{\infty} \frac{\underline{K}^j(y-x)^j}{j!} \tag{6.14}$$

The operations in each term of the series (6.14) are numerically possible and convergence of the infinite series can be established. Hence in principle the exponential of the matrix $\underline{K}(y-x)$ is computable arbitrarily accurately. It is easy to verify (just as in elementary calculus) that

$$\frac{d}{dy}\exp[\underline{K}(y-x)] = \exp[\underline{K}(y-x)]\underline{K} \tag{6.15}$$

Hence $\underline{M}(x,y)$, as given in (6.13), satisfies (6.8). When using the exponential form (6.13) for $\underline{M}(x,y)$, the mapping and group properties are immediately verified formally (e.g., $\exp[\underline{K}(z-y)]\exp[\underline{K}(y-x)] = \exp[\underline{K}(z-x)]$).

D. *Eigenmatrix Form of $M(x,y)$*

The operations in (6.14) would be considerably simpler if $\underline{K}$ were a diagonal matrix. Suppose for the moment we can reduce $\underline{K}$ to diagonal form $\underline{\kappa}$. That is, suppose we can find a $2q\times 2q$ invertible matrix $\underline{E}$ such that

$$\underline{K} = \underline{E}\,\underline{\kappa}\,\underline{E}^{-1} \qquad (2q\times 2q) \tag{6.16}$$

where $$\underline{\kappa} = \mathrm{diag}[\kappa_1,\ldots,\kappa_q,\kappa_{q+1},\ldots,\kappa_{2q}] \qquad (2q\times 2q) \tag{6.17}$$

Then directly from (6.14) we deduce that

$$\begin{aligned}\underline{M}(x,y) = \exp[\underline{K}(y-x)] &= \sum_{j=0}^{\infty} \underline{K}^j \frac{(y-x)^j}{j!}\\ &= \sum_{j=0}^{\infty} [\underline{E}\,\underline{\kappa}\,\underline{E}^{-1}]^j \frac{(y-x)^j}{j!}\\ &= \underline{E}\Big[\sum_{j=0}^{\infty} \frac{\underline{\kappa}^j (y-x)^j}{j!}\Big]\underline{E}^{-1}\\ &= \underline{E}\exp[\underline{\kappa}(y-x)]\underline{E}^{-1} \qquad (2q\times 2q)\end{aligned} \tag{6.18}$$

where $$\exp[\underline{\kappa}(y-x)] = \mathrm{diag}[\exp\kappa_1(y-x),\ldots,\exp\kappa_{2q}(y-x)] \qquad (2q\times 2q) \tag{6.19}$$

In this case, then, $\exp[\underline{K}(y-x)]$ can be evaluated numerically quite readily, provided we know the $2q \times 2q$ matrix $\underline{E}$ and the $2q$ numbers $\kappa_j, j = 1,\ldots,2q$. By (6.16) $\underline{E}$ and the κ_j are the eigenstructures of $\underline{K}$. That is, from (6.16), we have

$$\underline{K}\ \underline{E} = \underline{E}\ \underline{\kappa} \tag{6.20}$$

which requires that $\underline{E} = [\underline{e}_1 \cdots \underline{e}_q\ \underline{e}_{q+1} \cdots \underline{e}_{2q}]$ be thought of as a matrix made up of $2q \times 1$ vectors $\underline{e}_j$, $j = 1,\ldots,2q$, each of which satisfies the eigenvector equation

$$\underline{K}\ \underline{e}_j = \kappa_j\ \underline{e}_j \quad , \quad j = 1,\ldots,2q \tag{6.21}$$

From this we see that κ_j is the eigenvalue of $\underline{K}$ associated with the eigenvector $\underline{e}_j$ of $\underline{K}$. This fact about the $\underline{e}_j$ and κ_j is central to the present study. We shall next reapproach (6.20) from a more physical direction. This will allow us to see the eigenstructures $\underline{e}_j$ and κ_j as arising from the local reflectance and local transmittance matrices comprising $\underline{K}$ in (6.4).

7. PHYSICAL BASIS OF THE EIGENMATRIX REPRESENTATION OF THE FUNDAMENTAL SOLUTION

We return to the setting of (6.18) and (6.20) to provide a physical basis for these formulas. In particular we may ask: what is the physical basis for the diagonal matrix $\underline{\kappa}$ in (6.19) and for the vectors $\underline{e}_j$ forming $\underline{E}$? Further, what physical reason may be given for the invertibility of $\underline{E}$?

A. *A Natural Basis for the Radiance Amplitude Vectors*

If one plots the natural logarithm of N(y;u,v), for fixed u,v, as a function of depth y, in a deep homogeneous natural water body, one sees the curve become essentially straight from some depth y_o downward. There is a depth y_o for which this is uniformly true for all u,v, $u = 1,\ldots,m$; $v = 1,\ldots,2n$. Now, if the medium is *homogeneous* and *infinitely deep* and since (6.5) is a linear system, there is the intuitive suggestion that perhaps there may be some linear combination of the observable vectors $\underline{A}(y)$ that decays (or grows) *precisely* exponentially with depth y; and conversely, these purely exponentially decaying and exponentially growing functions may perhaps be linearly combined to yield $\underline{A}(y)$. To see where this leads, let us postulate the existence of 2q distinct exponential functions in y, over the range $x \le y \le z$, of the form

$$B_j^{\pm}(y) \equiv B_j^{\pm}(x) \exp[\kappa_j^{\pm}(y-x)] \qquad j = 1,\ldots,q \tag{7.1}$$

where $\kappa_j^{\pm}$, $j = 1,\ldots,q$ are 2q distinct real numbers.* They are just as general

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* Recall that y in the body of this study is optical depth, so that $y = \alpha\zeta$, where ζ is the associated geometric depth and α is the volume attenuation coefficient. Hence the $\kappa_j^{\pm}$ are dimensionless. They correspond to physical attenuation coefficients $k_j^{\pm} = \alpha\kappa_j^{\pm}$.

as the κ_j in (6.17), and in fact we can pair κ_j with κ_j^+ for $j = 1,\dots,q$, and κ_j with κ_j^- for $j = q+1,\dots,q$. Hence

$$\frac{d}{dy} B_j^{\pm}(y) = \kappa_j^{\pm} B_j(y) \qquad j = 1,\dots,q. \tag{7.2}$$

We can assemble these functions into the $1 \times q$ vectors corresponding to $\underline{A}^{\pm}(y)$, by writing

$$\underline{B}^{\pm}(y) \equiv [B_1^{\pm}(y),\dots,B_q^{\pm}(y)] \qquad (1\times q) \tag{7.3}$$

and into the $1\times 2q$ vector

$$\underline{B}(y) \equiv [\underline{B}^{+}(y), \underline{B}^{-}(y)] \qquad (1\times 2q) \tag{7.4}$$

corresponding to $\underline{A}(y)$. Thus (7.1) may be written as

$$\underline{B}^{\pm}(y) = \underline{B}^{\pm}(x) \exp[\underline{\kappa}^{\pm}(y-x)] \tag{7.5}$$

where

$$\underline{\kappa}^{\pm} \equiv \mathrm{diag}[\kappa_1^{\pm},\dots,\kappa_q^{\pm}] \qquad (q\times q) \tag{7.6}$$

and moreover,

$$\underline{B}(y) = \underline{B}(x) \exp[\underline{\kappa}(y-x)] \tag{7.7}$$

where

$$\underline{\kappa} = \mathrm{diag}[\underline{\kappa}^{+}, \underline{\kappa}^{-}] \qquad (2q\times 2q) \tag{7.8}$$

Furthermore (7.2) can be written as

$$\frac{d}{dy}\,\underline{B}^{\pm}(y) = \underline{B}^{\pm}(y)\,\underline{\kappa}^{\pm} \tag{7.9}$$

or more compactly still as

$$\frac{d}{dy}\,\underline{B}(y) = \underline{B}(y)\,\underline{\kappa} \tag{7.10}$$

Let us now return to (7.1) and observe that the 2q functions of y, $B_j^{\pm}(y)$, $j = 1,\dots,q$, over the depth range of y in X[x,z], are linearly independent. This follows at once from the fact that the $\kappa_j^{\pm}$ are pairwise distinct and a direct appeal to the definition of linear independence of functions over a common domain (cf. Courant, 1936, p. 439). The set of all linear combinations of the $B_j^{\pm}(y)$ therefore forms a 2q dimensional vector space of functions on X[x,z]. We assume that each radiance amplitude $A^{\pm}(y;u)$, for fixed u, is in such a space. (This assumption will be verified later.) Then we may write

$$A^{\pm}(y;u) = \sum_{j=1}^{q} B_j^{+}(y)\, f_j^{+\pm}(u) + \sum_{j=1}^{q} B_j^{-}(y)\, f_j^{-\pm}(u) \tag{7.11}$$

$$u = 1,\dots,q \quad , \quad x \le y \le z.$$

where the $f_j^{+\pm}(u)$ and $f_j^{-\pm}(u)$ are suitable coefficients, to be determined. Equation (7.11) may be placed into vector form if we write

$$\underline{f}_j^{+\pm} \equiv [f_j^{+\pm}(1),\ldots,f_j^{+\pm}(q)] \qquad (1\times q) \tag{7.12}$$

and

$$\underline{f}_j^{-\pm} \equiv [f_j^{-\pm}(1),\ldots,f_j^{-\pm}(q)] \qquad (1\times q) \tag{7.13}$$

over the index range $j = 1,\ldots,q$. Then (7.11) becomes

$$\underline{A}^{\pm}(y) = \sum_{j=1}^{q} B_j^{+}(y)\, \underline{f}_j^{+\pm} + \sum_{j=1}^{q} B_j^{-}(y)\, \underline{f}_j^{-\pm} \tag{7.14}$$

$$x \le y \le z$$

In matrix form, this is

$$[\underline{A}^{+}(y),\ \underline{A}^{-}(y)] = [\underline{B}^{+}(y),\ \underline{B}^{-}(y)] \begin{bmatrix} \underline{F}^{++} & \underline{F}^{+-} \\ \underline{F}^{-+} & \underline{F}^{--} \end{bmatrix} \tag{7.15}$$

or

$$\underline{A}(y) = \underline{B}(y)\ \underline{F} \tag{7.16}$$

where

$$\underline{F} = \begin{bmatrix} \underline{F}^{++} & \underline{F}^{+-} \\ \underline{F}^{-+} & \underline{F}^{--} \end{bmatrix}$$

and where

$$\underline{F}^{\pm\pm} = \begin{bmatrix} \underline{f}_1^{\pm\pm} \\ \vdots \\ \underline{f}_q^{\pm\pm} \end{bmatrix} \quad , \quad \underline{F}^{\pm\mp} = \begin{bmatrix} \underline{f}_1^{\pm\mp} \\ \vdots \\ \underline{f}_q^{\pm\mp} \end{bmatrix} \tag{7.17}$$

In this way we have physically motivated the $2q \times 2q$ depth independent mapping $\underline{F}$ of the exponential basis vector $\underline{B}(y)$ into the observable radiance amplitude vector $\underline{A}(y)$ at each y, $x \le y \le z$.

The preceding mapping can also be postulated to go in the reverse direction. Analogously to (7.11) we may now assume the existence of suitable coefficients $e_j^{+\pm}(u)$ and $e_j^{-\pm}(u)$, to be determined, so that

$$B_j^{\pm}(y) = \sum_{u=1}^{q} A^{+}(y;u)\, e_j^{+\pm}(u) + \sum_{u=1}^{q} A^{-}(y;u)\, e_j^{-\pm}(u) \tag{7.18}$$

$$j = 1,\ldots,q \quad , \quad x \le y \le z.$$

Following the pattern in (7.12) and (7.13) but noting the difference in running-index variables in each case, we can write, with $j = 1,\ldots,q$,

$$\begin{aligned} \underline{e}^{+\pm}(u) &\equiv [e_1^{+\pm}(u),\ldots,e_q^{+\pm}(u)] \\ \underline{e}^{-\pm}(u) &\equiv [e_1^{-\pm}(u),\ldots,e_q^{-\pm}(u)] \end{aligned} \tag{7.19}$$

Then (7.18) takes its vector form

$$\underline{B}^{\pm}(y) = \sum_{u=1}^{q} A^{+}(y;u)\, \underline{e}^{+\pm}(u) + \sum_{u=1}^{q} A^{-}(y;u)\, \underline{e}^{-\pm}(u) \tag{7.20}$$

In matrix form this is

$$[\underline{B}^+(y),\ \underline{B}^-(y)] = [\underline{A}^{\pm}(y),\ \underline{A}^-(y)] \begin{bmatrix} \underline{E}^{++} & \underline{E}^{+-} \\ \underline{E}^{-+} & \underline{E}^{--} \end{bmatrix} \tag{7.21}$$

i.e.,

$$\underline{B}(y) = \underline{A}(y)\,\underline{E} \tag{7.22}$$

where

$$\underline{E} = \begin{bmatrix} \underline{E}^{++} & \underline{E}^{+-} \\ \underline{E}^{-+} & \underline{E}^{--} \end{bmatrix}$$

and where

$$\underline{E}^{\pm\pm} = \begin{bmatrix} \underline{e}^{\pm\pm}(1) \\ \vdots \\ \underline{e}^{\pm\pm}(q) \end{bmatrix}, \quad \underline{E}^{\mp\pm} = \begin{bmatrix} \underline{e}^{\pm\mp}(1) \\ \vdots \\ \underline{e}^{\pm\mp}(q) \end{bmatrix}. \tag{7.23}$$

On comparing (7.10) and (7.22) we see that

$$\underline{F} = \underline{E}^{-1} \tag{7.24}$$

provided $\underline{E}^{-1}$ exists. Before looking into the matter of the existence of $\underline{E}^{-1}$, we derive from (7.16) and (7.22) the two results of immediate interest.

B. *The Eigenstructure Equation*

Let us suppose there is a mapping $\underline{E}$ (as in (7.22)) of an arbitrary solution vector $\underline{A}(y)$ of (6.5) into a vector of the form $\underline{B}(y)$ governed by (7.10). Then, on the one hand we have

$$\frac{d}{dy}\,\underline{B}(y) = \underline{B}(y)\underline{\kappa} = \underline{A}(y)\underline{\mathcal{E}}\;\underline{\kappa} \tag{7.25}$$

On the other hand, by (6.5) and (7.22)

$$\frac{d}{dy}\,[\underline{A}(y)\;\underline{\mathcal{E}}] = A(y)\underline{\mathcal{K}}\;\underline{E} = \frac{d}{dy}\,B(y) \tag{7.26}$$

Thus we find, on comparing (7.25) and (7.26), that $\underline{\mathcal{E}}$ and κ must necessarily satisfy the relation

$$\boxed{\underline{\mathcal{K}}\;\underline{\mathcal{E}} = \underline{\mathcal{E}}\;\underline{\kappa}} \tag{7.27}$$

which shows that $\underline{\mathcal{E}}$ and $\underline{\kappa}$ must be the eigenstructures of $\underline{\mathcal{K}}$. Thus $\underline{\mathcal{E}}$ and $\underline{\kappa}$ are determined solely by the local reflectances and transmittances of the medium. This is the first result.

C. *The Eigenrepresentation of* $\mathcal{M}(x,y)$

The connection between the fundamental solution $\underline{\mathcal{M}}(x,y)$ and the eigenstructures $\underline{\mathcal{E}}$ and $\underline{\kappa}$ of $\underline{\mathcal{K}}$ is the second result, and may be established as follows. By the mapping property (6.10), along with the connections (7.7) and (7.16), we have, for any $\underline{A}(x)$:

$$\underline{A}(x)\;\underline{\mathcal{M}}(x,y) = \underline{A}(y) = \underline{B}(y)\;\underline{\mathcal{F}} = \underline{B}(x)\;\exp[\underline{\kappa}(y-x)]\underline{\mathcal{F}} \tag{7.28}$$

By (7.22) for the case of $y = x$, we can replace $\underline{B}(x)$ by $\underline{A}(x)\underline{\mathcal{E}}$ to find

$$\underline{A}(x)\;\mathcal{M}(x,y) = \underline{A}(x)\;\underline{\mathcal{E}}\;\exp[\underline{\kappa}(y-x)]\underline{\mathcal{F}}$$

Since $\underline{A}(x)$ is arbitrary, we have the desired connection:

$$\boxed{\underline{M}(x,y) = \underline{E}\ \exp[\underline{\kappa}(y-x)]\underline{F}} \tag{7.29}$$

Thus we see the linear combination coefficients in (7.11) and (7.18) are connected directly with the eigenvectors of $\underline{K}$, while the growth and decay rates $\kappa_j^{\pm}$ of the $B_j^{\pm}(y)$ are the eigenvalues of $\underline{K}$. These are the desired physical interpretations of $\underline{E}$ and $\underline{\kappa}$ in (7.27).

We can now work backwards to construct the desired solution of (6.5): From knowledge of $\underline{K}$ we can solve the algebraic problem (7.27) to find the matrices $\underline{E}$ and $\underline{\kappa}$. From $\underline{\kappa}$ we can construct the $\underline{B}(y)$ by (7.7), and from $\underline{F} = \underline{E}^{-1}$ we can construct the amplitudes $\underline{A}(y) = \underline{B}(y)\underline{F}$. By (7.28), (7.29), and (6.18) we see that indeed $\underline{A}(y)$ is a solution of (6.5) with $\underline{K} = \underline{E}\ \underline{\kappa}\ \underline{E}^{-1}$. Thus (7.29) is the fundamental matrix solution yielding the desired radiance amplitude vectors $\underline{A}(y) = \underline{A}(x)\ \underline{M}(x,y)$ at each y in X[x,z], and associated initial amplitude $\underline{A}(x)$. After some further work on the transport formulation of this problem, we can translate the final results of the preceding steps into simple, elegant formulas (cf. (9.23) along with (16.2) and (16.3)).

8. PHYSICAL FEATURES OF THE EIGENMATRIX AND ITS EIGENVALUES

The decomposition of the radiance field into an upward (+) and a downward (−) set of flows imparts special properties to the eigenstructures $\underline{E}$ and $\underline{\kappa}$ of the system matrix $\underline{K}$ in (6.4). By exploiting our physical image of this two-flow decomposition, we can go considerably further than standard differential equation procedures in solving the system (6.5).

A. *Two-Flow Partition of E; first form*

Our first action will be to re-partition the matricial set of eigenvector solutions $\underline{e}_j$ in (6.20):

$$\underline{E} = [\underline{e}_1 \cdots \underline{e}_q \; \underline{e}_{q+1} \cdots \underline{e}_{2q}] \qquad (2q\times 2q) \qquad (8.1a)$$

associated with the eigenvalues

$$\kappa_1,\ldots,\kappa_q,\ \kappa_{q+1},\ldots,\kappa_{2q} \qquad (8.1b)$$

Going by the pattern (7.21), let us split each $\underline{e}_j$ in (8.1a) into two $q\times 1$ column vectors. Thus let us write

$$\begin{bmatrix} \underline{e}_j^{++} \\ \underline{e}_j^{-+} \end{bmatrix} \equiv \underline{e}_j \quad \text{when} \quad j = 1,\ldots,q \qquad (8.2)$$

and

$$\begin{bmatrix} \underline{e}_j^{+-} \\ \underline{e}_j^{--} \end{bmatrix} \equiv \underline{e}_j \quad \text{when} \quad j = q+1,\ldots,2q \qquad (8.3)$$

This is simply another way of arranging the coefficients $e_j^{\pm\pm}(u)$ and $e_j^{\pm\mp}(u)$ in (7.18). The reason for this rearrangement will now become clear.

B. *Reversal Property of Eigenvectors and Eigenvalues*

Suppose $\underline{e}_j$ is a $2q\times 1$ vector of the form (8.2) with eigenvalue κ_j, $j = 1,\ldots,q$. The *reverse* $\underline{e}_j^R$ of $\underline{e}_j$ is defined as

$$\underline{e}_j^R \equiv \begin{bmatrix} \underline{e}_j^{-+} \\ \underline{e}_j^{++} \end{bmatrix} = \begin{bmatrix} \underline{U}_q & \underline{I}_q \\ \underline{I}_q & \underline{U}_q \end{bmatrix} \begin{bmatrix} \underline{e}_j^{++} \\ \underline{e}_j^{-+} \end{bmatrix}$$

$$\equiv \underline{Q}\ \underline{e}_j \tag{8.4}$$

Here $\underline{Q}$ is the $2q\times 2q$ *reversal* matrix. Next observe by straightforward computation that $\underline{Q}$ has the properties

$$\underline{Q}^2 = \underline{I}_{2q} \tag{8.5}$$

and

$$\underline{Q}\ \underline{K} = -\underline{K}\ \underline{Q} \tag{8.6}$$

In view of (8.5), this last equation may be written

$$\underline{Q}\ \underline{K}\ \underline{Q} = -\underline{K} \tag{8.7}$$

Now let $\underline{e}_j$ be an eigenvector of $\underline{K}$ with associated eigenvalue κ_j. Then by definition

$$\underline{K}\ \underline{e}_j = \kappa_j\ \underline{e}_j \qquad j = 1,\ldots,q \tag{8.8}$$

Multiplying (8.8) by $\underline{Q}$ on the left and using (8.5) we find

$$\underline{Q}\,\underline{K}\,\underline{Q}^2\,\underline{e}_j = \kappa_j\,\underline{Q}\underline{e}_j$$

which, by (8.7) and (8.4) reduces to

$$\underline{K}\,\underline{e}_j^R = -\kappa_j\,\underline{e}_j^R \tag{8.9}$$

$$j = 1,\ldots,q.$$

We conclude that if $(\underline{e}_j, \kappa_j)$ is an eigenpair of $\underline{K}$, then so also is $(\underline{e}_j^R, -\kappa_j)$, $j = 1,\ldots,q$. From this we see that the eigenvalues κ_j of $\underline{K}$ come in signed pairs κ_j, $-\kappa_j$, $j = 1,\ldots,q$. If we re-index these eigenvalues so that in (8.1b), $\kappa_{j+q} = -\kappa_j$ for $j = 1,\ldots,q$, then it follows that in (8.1a), $\underline{e}_{j+q} = \underline{e}_j^R$ for $j = 1,\ldots,q$. Hence in (8.2) and (8.3) we deduce that, for $j = 1,\ldots,q$

$$\underline{e}_j^{++} = \underline{e}_j^{--} \equiv \underline{e}_j^{+} \tag{8.10a}$$

$$\underline{e}_j^{-+} = \underline{e}_j^{+-} \equiv \underline{e}_j^{-} \tag{8.10b}$$

In this way we see how the isotropy of the volume scattering function allows a simplification of the double-direction superscripts of the eigenvectors.

C. *Two-Flow Partition of $\underline{E}$; second form*

By (8.10), we may now write $\underline{E}$ in (8.1a) as

$$\underline{E} \equiv \begin{bmatrix} \underline{E}^{+} & \underline{E}^{-} \\ \underline{E}^{-} & \underline{E}^{+} \end{bmatrix} \tag{8.11}$$

In other words, the reverse symmetry of $\underline{K}$ implies that in (7.22),

$$\underline{E}^{++} = \underline{E}^{--} \equiv \underline{E}^{+} \tag{8.12a}$$

$$\underline{E}^{-+} = \underline{E}^{+-} \equiv \underline{E}^{-} \tag{8.12b}$$

The specific display of the eigenvector partitions in $\underline{E}$ is then

$$\underline{E} = \left[\begin{array}{ccc|ccc} \underline{e}_1^{+} & \cdots & \underline{e}_1^{+} & \underline{e}_1^{-} & \cdots & \underline{e}_q^{-} \\ \hline e_1^{-} & \cdots & e_q^{-} & e_1^{+} & \cdots & e_q^{+} \end{array}\right] = \left[\begin{array}{c|c} \underline{E}^{+} & \underline{E}^{-} \\ \hline \underline{E}^{-} & \underline{E}^{+} \end{array}\right] \tag{8.13}$$

where

$$\underline{e}_j^{\pm} \equiv [e_j^{\pm}(1),\ldots,e_j^{\pm}(q)]^T \quad , \quad j = 1,\ldots,q \qquad (q\times 1)$$

and where "T" denotes transpose, so each $\underline{e}_j^{\pm}$ is $q\times 1$.

D. *Two Basic Physical Features of the Radiance Field*

We now use some of our intuition about light fields in natural and laboratory waters to infer informally some physically plausible properties about the eigenstructures of the system matrix $\underline{K}$ in (6.4), as are expected to hold in such media. In optically deep source-free homogeneous media $X[x,z]$ with positive absorption coefficient, $a > 0$ (at some arbitrary fixed wavelength), we expect that

(i) Starting at level x with input $A^-(x,u)$ and zero input $A^+(z,u)$, $A^-(y;u)$ for each fixed u eventually decays exponentially with increasing depth y. Conversely, starting at level z, with input $A^+(z;u)$ and zero input $A^-(x;u)$, $A^+(y;u)$ for each fixed u eventually decays with decreasing depth y.

(ii) Each of the + and - modes of decay in (i) has q degrees of freedom at and near the respective initial boundary. [For example, the sun can be systematically raised above the horizon while we are at level x. For each sun position, $\underline{A}^-(x)$ is then seen to take on a particular orientation in its q dimensional vector space. Hence the full q-dimensionality of $\underline{A}(y)$ for y near x must be employed to cover this wide range of incident radiances.]

Now without loss of generality we may arrange the non-negative members of the set of 2q eigenvalues $\pm\kappa_j$, $j = 1,\dots,q$ into ascending order: $\kappa_1 \le \kappa_2 \le \cdots \le \kappa_q$. From (i) we deduce that κ_1 must be positive: $\kappa_1 > 0$. From (ii) we deduce that the q eigenvalues κ_j must be distinct, so that $0 < \kappa_1 < \kappa_2 < \cdots < \kappa_q$. Next, from linear algebra we know that the pairwise distinctness of the 2q eigenvalues κ_j of $\underline{K}$ implies the linear independence of the set of 2q eigenvectors $\underline{e}_j$ of $\underline{K}$ (cf. Franklin, 1968, p. 73), so that $\underline{E}$ has an inverse $\underline{E}^{-1} \equiv \underline{F}$. Statement (ii) actually implies a stronger property of $\underline{E}$, by virtue of (7.20): the set of vectors $\underline{e}_1^+,\dots,\underline{e}_q^+$ and the set $\underline{e}_1^-,\dots,\underline{e}_q^-$ are each linearly independent. Hence $(\underline{E}^+)^{-1}$ and $(\underline{E}^-)^{-1}$ must exist on physical grounds.

While the preceding properties (i), (ii) and their consequences are not offered as mathematical theorems, the reader may perhaps agree that for a real plane-parallel homogeneous optical medium with $a > 0$ drawn at random, the probability is zero that the preceding assertions about $\underline{E}$ and $\underline{K}$ are not true. Indeed, we can reverse the matter as follows. We can say that a choice of $\alpha(y,\lambda)$ and $\sigma(y;\xi';\xi,\lambda)$ for a homogeneous medium is *realistic* provided $\underline{\hat{\rho}} \neq \underline{0}$ and provided the associated eigenvalues $\pm\kappa_j$, $j = 1,\ldots,q$ are such that $0 < \kappa_1 < \cdots < \kappa_q$, for all wavelengths, all azimuthal modes, and all choices of q. The resultant model is then a *realistic model*, by definition. It would appear that a necessary and sufficient condition for a model to be realistic is that $\alpha(y,\lambda)$ and $\sigma(y;\xi';\xi,\lambda)$ yield the inequalities $s(y,\lambda) > 0$ and $\alpha(y,\lambda) - s(y,\lambda) = a(y,\lambda) > 0$ for all y and λ.

E. *Inversion of $\underline{E}$ by Partitioning*

It will be useful to find explicit expressions for the inverse $\underline{F}$ of $\underline{E}$ defined in (7.24). Our discussion in paragraph D above showed that on physical grounds the existence of $\underline{E}^{-1}$ is practically certain. Let us partition the $2q \times 2q$ matrix $\underline{F}$ analogously to $\underline{E}$ in (8.11), so that

$$\underline{F} = \left[\begin{array}{c|c} \underline{F}^{+} & \underline{F}^{-} \\ \hline \underline{F}^{-} & \underline{F}^{+} \end{array}\right] = \left[\begin{array}{c|c} \underline{f}_1^{+} & \underline{f}_1^{-} \\ \vdots & \vdots \\ \underline{f}_q^{+} & \underline{f}_q^{-} \\ \hline \underline{f}_1^{-} & \underline{f}_1^{+} \\ \vdots & \vdots \\ \underline{f}_q^{-} & \underline{f}_q^{+} \end{array}\right] \qquad (2q\times 2q) \tag{8.14}$$

where $\quad \underline{f}_j^{\pm} = [f_j^{\pm}(1),\ldots,f_j^{\pm}(q)]$

Then by definition, requiring

$$\begin{bmatrix} \underline{F}^+ & \underline{F}^- \\ \underline{F}^- & \underline{F}^+ \end{bmatrix} \begin{bmatrix} \underline{E}^+ & \underline{E}^- \\ \underline{E}^- & \underline{E}^+ \end{bmatrix} = \begin{bmatrix} \underline{I}_q & \underline{0}_q \\ \underline{0}_q & \underline{I}_q \end{bmatrix} \tag{8.15}$$

yields

$$\underline{F}^+\underline{E}^+ + \underline{F}^-\underline{E}^- = \underline{I}_q \tag{8.16}$$

$$\underline{F}^-\underline{E}^+ + \underline{F}^+\underline{E}^- = \underline{0}_q \tag{8.17}$$

whence

$$\underline{F}^- = - \underline{F}^+\underline{E}^-(\underline{E}^+)^{-1} \tag{8.18}$$

and

$$\underline{F}^+ = [\underline{E}^+ - \underline{E}^-(\underline{E}^+)^{-1}\,\underline{E}^-]^{-1} \tag{8.19}$$

Alternately, by definition, requiring

$$\begin{bmatrix} \underline{E}^+ & \underline{E}^- \\ \underline{E}^- & \underline{E}^+ \end{bmatrix} \begin{bmatrix} \underline{F}^+ & \underline{F}^- \\ \underline{F}^- & \underline{F}^+ \end{bmatrix} = \begin{bmatrix} \underline{I}_q & \underline{0}_q \\ \underline{0}_q & \underline{I}_q \end{bmatrix} \qquad (8.20)\ (8.21)$$

yields

$$\underline{E}^+\underline{F}^+ + \underline{E}^-\underline{F}^- = \underline{I}_q \tag{8.22}$$

$$\underline{E}^-\underline{F}^+ + \underline{E}^+\underline{F}^- = \underline{0}_q \tag{8.23}$$

whence

$$\underline{F}^+ = -(\underline{E}^-)^{-1}\,\underline{E}^+\underline{F}^- \tag{8.24}$$

and

$$\underline{F}^- = [\underline{E}^- - \underline{E}^+(\underline{E}^-)^{-1}\,\underline{E}^+]^{-1} \tag{8.25}$$

Therefore, to find $\underline{E}^{-1}$, i.e. to find $\underline{F}^{\pm}$, we may invert the smaller matrices $\underline{E}^{\pm}$ and their algebraic combinations as indicated in (8.18) and (8.19) or in (8.24) and (8.25).

9. REPRESENTATION OF THE FUNDAMENTAL MATRIX BY EIGENSTRUCTURES

A. *Decomposing the Fundamental Matrix*

The fundamental matrix $M(x,y)$ in the mapping rule (6.10) and in (7.29) can be given a more useful representation provided we partition it in the form

$$\underline{M}(x,y) \equiv \begin{bmatrix} \underline{M}_{++}(x,y) & \underline{M}_{+-}(x,y) \\ \underline{M}_{-+}(x,y) & \underline{M}_{--}(x,y) \end{bmatrix} \qquad (2q\times 2q) \tag{9.1}$$

Then by (7.29) and (8.13) we have

$$\begin{bmatrix} \underline{M}_{++}(x,y) & \underline{M}_{+-}(x,y) \\ \underline{M}_{-+}(x,y) & \underline{M}_{--}(x,y) \end{bmatrix} = \begin{bmatrix} \underline{E}^{+} & \underline{E}^{-} \\ \underline{E}^{-} & \underline{E}^{+} \end{bmatrix} \begin{bmatrix} e^{\underline{\kappa}(y-x)} & \underline{0}_q \\ \underline{0}_q & e^{-\underline{\kappa}(y-x)} \end{bmatrix} \begin{bmatrix} \underline{F}^{+} & \underline{F}^{-} \\ \underline{F}^{-} & \underline{F}^{+} \end{bmatrix} \tag{9.2}$$

where $\underline{\kappa} = \mathrm{diag}[\kappa_1,\ldots,\kappa_q]$, and where the κ_j now have their indexing as in §8D. From this, we find the representations of the $q\times q$ submatrices of $\underline{M}(x,y)$:

$$\underline{M}_{++}(x,y) = \underline{E}^{+}\, e^{\underline{\kappa}(y-x)}\, \underline{F}^{+} + \underline{E}^{-}\, e^{-\underline{\kappa}(y-x)}\, \underline{F}^{-} \tag{9.3}$$

$$\underline{M}_{-+}(x,y) = \underline{E}^{-}\, e^{\underline{\kappa}(y-x)}\, \underline{F}^{+} + \underline{E}^{+}\, e^{-\underline{\kappa}(y-x)}\, \underline{F}^{-} \tag{9.4}$$

$$\underline{M}_{+-}(x,y) = \underline{E}^{+}\, e^{\underline{\kappa}(y-x)}\, \underline{F}^{-} + \underline{E}^{-}\, e^{-\underline{\kappa}(y-x)}\, \underline{F}^{+} \tag{9.5}$$

$$\underline{M}_{--}(x,y) = \underline{E}^{-}\, e^{\underline{\kappa}(y-x)}\, \underline{F}^{-} + \underline{E}^{+}\, e^{-\underline{\kappa}(y-x)}\, \underline{F}^{+} \tag{9.6}$$

Clearly

$$\underline{M}_{++}(x,x) = \underline{M}_{--}(x,x) = \underline{I}_q \tag{9.7}$$

$$\underline{M}_{+-}(x,x) = \underline{M}_{-+}(x,x) = \underline{0}_q \tag{9.8}$$

Observe that $\underline{M}(x,y)$ depends only on the difference y-x of initial and final depths in the homogeneous medium X[x,z]. For small depth differences y-x, the submatrices in (9.3)-(9.6) are, to first order in (y-x),

$$\underline{M}_{++}(x,y) = \underline{I}_q - \hat{\underline{\tau}}(y-x) \tag{9.3a}$$

$$\underline{M}_{-+}(x,y) = - \hat{\underline{\rho}}(y-x) \tag{9.4a}$$

$$\underline{M}_{+-}(x,y) = \hat{\underline{\rho}}(y-x) \tag{9.5a}$$

$$\underline{M}_{--}(x,y) = \underline{I}_q + \hat{\underline{\tau}}(y-x) \tag{9.6a}$$

These follow from the fact that, for small y-x, (6.14) is, to first order,

$$\underline{M}(x,y) = \underline{I}_{2q} + \underline{K}(y-x)$$

B. *Interchange Rule for* $\underline{M}(x,y)$

In a medium such as the present one where we have isotropy of scattering and homogeneity of inherent properties with depth, we should expect some useful symmetries of $\underline{M}(x,y)$. For example, isotropy has reduced our considerations to $\underline{E}^{+}$ and $\underline{E}^{-}$ instead of the four matrices $\underline{E}^{++},\ldots,\underline{E}^{--}$.

A considerable amount of computation, however, is obviated by observing (cf. (6.12)) that $\underline{M}^{-1}(x,y) = \underline{M}(y,x)$. The latter matrix is readily evaluated by observing from (9.3)-(9.6) that its four submatrices are related to those of $\underline{M}(x,y)$ by

$$\underline{M}_{++}(y,x) = \underline{M}_{--}(x,y) \tag{9.9}$$

$$\underline{M}_{-+}(y,x) = \underline{M}_{+-}(x,y) \tag{9.10}$$

$$\underline{M}_{+-}(y,x) = \underline{M}_{-+}(x,y) \tag{9.11}$$

$$\underline{M}_{--}(y,x) = \underline{M}_{++}(x,y) \tag{9.12}$$

These four statements constitute the *interchange rule* for $\underline{M}(x,y)$ on X[x,z]. It holds quite generally and does not depend on isotropy or homogeneity. However, when homogeneity is taken into account, we can find $\underline{M}(z,y)$ without further computation beyond $\underline{M}(x,y)$. The matrix $\underline{M}(z,y)$ is used for upward evolving light fields, while $\underline{M}(x,y)$ is used for downward evolving light fields, as we shall see, below. As for evaluating $\underline{M}(z,y)$, suppose $z-y = y-x$, then $\underline{M}(z,y) = \underline{M}^{-1}(y,z) = M^{-1}(x,y) = \underline{M}(y,x)$. The latter follows from $\underline{M}(x,y)$ by the interchange rule.

C. *The Downward and Upward Evolving Radiance Amplitudes*

From (6.10) and (9.1) we can write the mapping rule as

$$\underline{A}^{+}(y) = \underline{A}^{+}(x)\ \underline{M}_{++}(x,y) + \underline{A}^{-}(x)\ \underline{M}_{-+}(x,y) \tag{9.13}$$

$$\underline{A}^{-}(y) = \underline{A}^{+}(x)\ \underline{M}_{+-}(x,y) + \underline{A}^{-}(x)\ \underline{M}_{--}(x,y) \tag{9.14}$$

This is for the downward evolving light field starting from level x in X[x,z], $x < z$. This may be reduced to the numerical level by explicitly writing out the components of $\underline{M}_{++}(x,y),\ldots,\underline{M}_{--}(x,y)$. Thus from (9.3)-(9.6) and (8.13) we have

$$\underline{M}_{++}(x,y) = \sum_{j=1}^{q} \underline{e}_j^+ \, e^{\kappa_j(y-x)} \, \underline{f}_j^+ + \sum_{j=1}^{q} \underline{e}_j^- \, e^{-\kappa_j(y-x)} \, \underline{f}_j^- \tag{9.15}$$

$$\underline{M}_{-+}(x,y) = \sum_{j=1}^{q} \underline{e}_j^- \, e^{\kappa_j(y-x)} \, \underline{f}_j^+ + \sum_{j=1}^{q} \underline{e}_j^+ \, e^{-\kappa_j(y-x)} \, \underline{f}_j^- \tag{9.16}$$

$$\underline{M}_{+-}(x,y) = \sum_{j=1}^{q} \underline{e}_j^+ \, e^{\kappa_j(y-x)} \, \underline{f}_j^- + \sum_{j=1}^{q} \underline{e}_j^- \, e^{-\kappa_j(y-x)} \, \underline{f}_j^+ \tag{9.17}$$

$$\underline{M}_{--}(x,y) = \sum_{j=1}^{q} \underline{e}_j^- \, e^{\kappa_j(y-x)} \, \underline{f}_j^- + \sum_{j=1}^{q} \underline{e}_j^- \, e^{-\kappa_j(y-x)} \, \underline{f}_j^+ \tag{9.18}$$

Next, the initial condition for the radiance amplitude at level x is handled by writing, for $j = 1,\ldots,q$,

$$a_j^+(x) \equiv \underline{A}^+(x) \, \underline{e}_j^+ + \underline{A}^-(x) \, \underline{e}_j^- \qquad (= \underline{A}(x) \, \underline{e}_j) \tag{9.19}$$

and

$$a_j^-(x) \equiv \underline{A}^+(x) \, \underline{e}_j^- + \underline{A}^-(x) \, \underline{e}_j^+ \qquad (= \underline{A}(x) \, \underline{e}_j^R) \tag{9.20}$$

Observe that $\underline{A}^{\pm}(x)$, being a $1{\times}q$ matrix, and $\underline{e}_j^{\pm}$, being a $q{\times}1$ matrix, imply that $a_j^{\pm}(x)$ are scalars of the form

$$a_j^+(x) = \sum_{u=1}^{q} [A^+(x;u) \, e_j^+(u) + A^-(x;u) \, e_j^-(u)] \tag{9.21}$$

and

$$a_j^-(x) = \sum_{u=1}^{q} [A^+(x;u) \, e_j^-(u) + A^-(x;u) \, e_j^+(u)] \tag{9.22}$$

for $j = 1,\ldots,q$.

Thus (9.13) and (9.14) in scalar form are

$$\boxed{\begin{array}{l} A^{\pm}(y;u) = \sum_{j=1}^{q} a_j^{+}(x)\, e^{\kappa_j(y-x)}\, f_j^{\pm}(u) + \sum_{j=1}^{q} a_j^{-}(x)\, e^{-\kappa_j(y-x)}\, f_j^{\mp}(u) \\ \\ u = 1,\ldots,q. \\ x \le y \le z. \end{array}} \tag{9.23}$$

which describes the downward evolving light field starting at level x in X[x,z]. This may be used repeatedly in a given medium by simply changing the initial amplitudes $a_j^{\pm}(x)$, $j = 1,\ldots,q$. Recall that (9.23) holds for a particular sine or cosine amplitude $A_p(y;u;\ell)$ with ℓ a particular azimuthal mode index; $p = 1,2$ and $\ell = 0,\ldots,n$, and with $q = m-1$ or m.

The upward evolving light field starting at level z in X[x,z] may be found similarly. Thus from the mapping property for $\underline{M}(z,y)$,

$$\underline{A}^{+}(y) = \underline{A}^{+}(z)\, \underline{M}_{++}(z,y) + \underline{A}^{-}(z)\, \underline{M}_{-+}(z,y) \tag{9.24}$$

$$\underline{A}^{-}(y) = \underline{A}^{+}(z)\, \underline{M}_{+-}(z,y) + \underline{A}^{-}(z)\, \underline{M}_{--}(z,y) \tag{9.25}$$

If in (9.13) and (9.14) we replace all occurrences of "x" with "z" we obtain (9.24) and (9.25). Hence the present counterpart for (9.23) is found by replacing all occurrences of "x" with "z" in (9.23). Equation (9.23) accordingly represents the general solution of the radiance amplitude equation (6.5). For numerical values we would need the κ_j and the $\underline{e}_j$, along with the initial amplitudes $\underline{A}(x)$. If $\underline{A}(x)$ is not given empirically, then it must be found theoretically from the initial radiance amplitude $\underline{A}(a)$ incident at level a on the air-water surface. For this we need the global interaction principles and various of their consequent laws governing the reflectances and transmittances of sub-layers of X[x,z].

10. GLOBAL INTERACTION PRINCIPLES

The global interaction principles allow us to correctly include boundary conditions into the fundamental solution procedure.

By following the developments in §6c of Mobley and Preisendorfer (1988), or in §7.4 of H.O., Vol. IV, we may derive from the fundamental solution two sets of global interaction principles for the radiance amplitudes in X[x,z]. First the *downward evolution* set: for a subslab X[x,y] of X[x,z], $x \le y \le z$, we have

$$\underline{A}^{+}(x) = \underline{A}^{+}(y)\ \underline{T}(y,x) + \underline{A}^{-}(x)\ \underline{R}(x,y) \tag{10.1}$$

$$\underline{A}^{-}(y) = \underline{A}^{+}(y)\ \underline{R}(y,x) + \underline{A}^{-}(x)\ \underline{T}(x,y) \tag{10.2}$$

These statements, while written explicitly for inside the water body part X[x,z] of X[a,b] = X[a,x] U X[x,z] U X[z,b], actually can be phrased for any subslab of X[a,b] by replacing x and y by other depth variables in the range [a,b]. For subslabs X[x,y] the water body itself, we can evaluate the $\underline{R}$ and $\underline{T}$ matrices as follows, using the fundamental matrix:

$$\underline{R}(y,x) = \underline{M}_{++}^{-1}(x,y)\ \underline{M}_{+-}(x,y) \tag{10.3}$$

$$\underline{T}(x,y) = \underline{M}_{--}(x,y) - \underline{M}_{+-}(x,y)\ \underline{M}_{++}^{-1}(x,y)\ \underline{M}_{+-}(x,y) \tag{10.4}$$

and

$$\underline{T}(y,x) = \underline{M}^{-1}_{++}(x,y) \tag{10.5}$$

$$\underline{R}(x,y) = -\underline{M}_{-+}(x,y)\, \underline{M}^{-1}_{++}(x,y) \tag{10.6}$$

Then the *upward evolution* set for a subslab X[y,z] of X[x,z], $x \le y \le z$, is

$$\boxed{\begin{aligned} \underline{A}^{+}(y) &= \underline{A}^{+}(z)\, \underline{T}(z,y) + \underline{A}^{-}(y)\, \underline{R}(y,z) && (10.7) \\ \underline{A}^{-}(z) &= \underline{A}^{+}(z)\, \underline{R}(z,y) + \underline{A}^{-}(y)\, \underline{T}(y,z) && (10.8) \end{aligned}}$$

Once again, these statements can be extended to any subslab of X[a,b] by suitable replacement of y,z in (10.7) and (10.8) by other depth variables. In the case of the water body X[x,z] itself, the $\underline{R}$ and $\underline{T}$ matrices are given by

$$\underline{R}(y,z) = \underline{M}^{-1}_{--}(z,y)\, \underline{M}_{-+}(z,y) \tag{10.9}$$

$$\underline{T}(z,y) = \underline{M}_{++}(z,y) - \underline{M}_{+-}(z,y)\, \underline{M}^{-1}_{--}(z,y)\, \underline{M}_{-+}(z,y) \tag{10.10}$$

and

$$\underline{T}(y,z) = \underline{M}^{-1}_{--}(z,y) \tag{10.11}$$

$$\underline{R}(z,y) = -\underline{M}_{+-}(z,y)\, \underline{M}^{-1}_{--}(z,y) \tag{10.12}$$

As a consequence of the homogeneity and isotropy of the medium X[x,z], we find from the interchange rule and the above *M*-representations of $\underline{R}$ and $\underline{T}$ that

$$\underline{R}(x,y) = \underline{R}(y,x) \qquad (= R(0,y-x)) \tag{10.13}$$

$$\underline{T}(x,y) = \underline{T}(y,x) \qquad (= T(0,y-x)) \tag{10.14}$$

In other words there is no polarity of the medium (cf. Preisendorfer, 1965, p. 216). This cuts in half the number of $\underline{R}$ and $\underline{T}$ matrices needed to find the light field in X[x,z].

We observe that when y-x is small, the various reflectances and transmittances in (10.3)-(10.6) and in (10.9)-(10.12) take the following forms, to first order in y-x:

$$\underline{R}(y,x) = \underline{R}(x,y) = \underline{\hat{\rho}}(y-x) \tag{10.15}$$

$$\underline{T}(y,x) = \underline{T}(x,y) = \underline{I}_q + \underline{\hat{\tau}}(y-x) \tag{10.16}$$

These follow at once by using (9.3a-9.6a) in the cited formulas and, after reducing them algebraically, retaining only terms to first order in y-x.

11. THE R-INFINITY FORMULA AND SOME APPLICATIONS

The reflectance matrix for an infinitely deep homogeneous layer $X[x,\infty]$ plays a central role in the present study of the light field. We shall now derive an expression for this reflectance and draw some conclusions about the light field.

Now, from (10.6) we have an expression for $\underline{R}(x,y)$, the reflectance of the finitely deep layer $X[x,y]$ in $X[x,\infty]$. On letting $y\to\infty$, using (9.3), (9.4) and recalling that $\kappa_1>0$, we have

$$\underline{R}_\infty \equiv \lim_{y\to\infty} \underline{R}(x,y) = \lim_{y\to\infty} -\underline{M}_{-+}(x,y)\, \underline{M}_{++}^{-1}(x,y) = -\underline{E}^-(\underline{E}^+)^{-1} \tag{11.1}$$

and in like manner, from (10.3), (9.3), and (9.5), we find

$$\begin{aligned}\lim_{y\to\infty} \underline{R}(y,x) = \lim_{y\to\infty} \underline{M}_{++}^{-1}(x,y)\, \underline{M}_{+-}(x,y) &= (\underline{F}^+)^{-1}\underline{F}^- \\ &= \underline{R}_\infty\end{aligned} \tag{11.2}$$

where the last equality comes from (8.17). Hence

$$\boxed{\underline{R}_\infty = -\underline{E}^-(\underline{E}^+)^{-1} = (\underline{F}^+)^{-1}\,\underline{F}^- \qquad (q\times q)} \tag{11.3}$$

The physical significance of $\underline{R}_\infty$, $\underline{E}^\pm$ and $\underline{F}^\pm$ comes out on rearranging (11.3) into the forms

$$\underline{E}^- = -\underline{R}_\infty\, \underline{E}^+ \tag{11.4}$$

or

$$\underline{F}^- = \underline{F}^+\, \underline{R}_\infty \tag{11.5}$$

In vector form these read

$$\underline{e}_j^- = -\underline{R}_\infty \, \underline{e}_j^+ \tag{11.4a}$$

$$\underline{f}_j^- = \underline{f}_j^+ \, \underline{R}_\infty \tag{11.5a}$$

for $j = 1,\ldots,q$. Thus we see that $\underline{R}_\infty$ maps the "disembodied" flow $\underline{\mathcal{F}}^+$ into $\underline{\mathcal{F}}^-$, and also $\underline{\mathcal{E}}^+$ into $-\underline{\mathcal{E}}^-$. Of course the main interpretation of $\underline{R}_\infty$ is obtained from (10.7) on letting $z\to\infty$. It is clear by homogeneity and from (10.5) that as $z\to\infty$, $\underline{T}(z,y) \to \underline{0}_q$, and so in $X[x,\infty]$, we have

$$\underline{A}^+(y) = \underline{A}^-(y) \, \underline{R}_\infty \tag{11.6}$$

$$x \le y < \infty$$

If we explicitly identify the entries of $\underline{R}_\infty$ via

$$\underline{R}_\infty = \begin{bmatrix} R_\infty(1,1) & \cdots & R_\infty(1,q) \\ \vdots & & \vdots \\ R_\infty(q,1) & \cdots & R_\infty(q,q) \end{bmatrix} \tag{11.7}$$

then (11.4a) and (11.5a) state

$$e_j^-(r) = -\sum_{u=1}^{q} R_\infty(r,u) \, e_j^+(u) \quad , \quad r = 1,\ldots,q \tag{11.8}$$

$$f_j^-(u) = \sum_{r=1}^{q} f_j^+(r) \, R_\infty(r,u) \quad , \quad u = 1,\ldots,q \tag{11.9}$$

where we have used the component relation defined in (8.13) and (8.14).

Another useful set of relations comes from opening up (7.27):

$$\begin{bmatrix} -\underline{\hat{\tau}} & \underline{\hat{\rho}} \\ -\underline{\hat{\rho}} & \underline{\hat{\tau}} \end{bmatrix} \begin{bmatrix} \underline{E}^{+} & \underline{E}^{-} \\ \underline{E}^{-} & \underline{E}^{+} \end{bmatrix} = \begin{bmatrix} \underline{E}^{+} & \underline{E}^{-} \\ \underline{E}^{-} & \underline{E}^{+} \end{bmatrix} \begin{bmatrix} \underline{\kappa} & \underline{0} \\ \underline{0} & -\underline{\kappa} \end{bmatrix} \tag{11.10}$$

where $\underline{\kappa} = \mathrm{diag}[\kappa_1,\ldots,\kappa_q]$. Thus we obtain two independent statements. The first is

$$-\underline{\hat{\tau}}\,\underline{E}^{+} + \underline{\hat{\rho}}\,\underline{E}^{-} = \underline{E}^{+}\,\underline{\kappa} \tag{11.11}$$

whence

$$\boxed{\underline{\hat{\rho}}\,\underline{R}_{\infty} + \underline{\hat{\tau}} = -\underline{E}^{+}\,\underline{\kappa}\,(\underline{E}^{+})^{-1}} \tag{11.12}$$

The second is

$$-\underline{\hat{\tau}}\,\underline{E}^{-} + \underline{\hat{\rho}}\,\underline{E}^{+} = -\underline{E}^{-}\,\underline{\kappa} \tag{11.13}$$

whence

$$\boxed{\underline{\hat{\tau}}\,\underline{R}_{\infty} + \underline{\hat{\rho}} = -\underline{E}^{-}\,\underline{\kappa}\,(\underline{E}^{+})^{-1}} \tag{11.14}$$

Eliminating $\underline{\kappa}$ from (11.12) and (11.14), by using (11.1), we find

$$\underline{R}_{\infty}[\underline{\hat{\rho}}\,\underline{R}_{\infty} + \underline{\hat{\tau}}] + [\underline{\hat{\tau}}\,\underline{R}_{\infty} + \underline{\hat{\rho}}] = \underline{0}_q \tag{11.15}$$

or equivalently

$$\boxed{\underline{R}_{\infty}\,\underline{\hat{\rho}}\,\underline{R}_{\infty} + (\underline{R}_{\infty}\,\underline{\hat{\tau}} + \underline{\hat{\tau}}\,\underline{R}_{\infty}) + \underline{\hat{\rho}} = \underline{0}_q} \tag{11.16}$$

This formula shows how $\underline{R}_\infty$ is determined as the solution of a matrix quadratic equation for $\underline{R}_\infty$ in terms of the local reflectance and transmittance matrices.

The $\underline{R}_\infty$ formula (11.3) may be used to simplify the expressions for $\underline{F}^{\pm}$ in (8.19) and (8.25):

$$\underline{F}^{+} = [\underline{E}^{+} + \underline{R}_\infty \underline{E}^{-}]^{-1} = (\underline{E}^{+})^{-1}[\underline{I} - R_\infty^2]^{-1} \tag{11.17}$$

$$\underline{F}^{-} = [\underline{E}^{-} + \underline{R}_\infty^{-1} \underline{E}^{+}]^{-1} = (\underline{E}^{+})^{-1}[\underline{I} - R_\infty^2]^{-1} \underline{R}_\infty \tag{11.18}$$

These results allow the factoring of $\underline{M}(x,y)$ into a product of four basic matrices. Thus (9.2) becomes

$$\underline{M}(x,y) =$$

$$\begin{bmatrix} \underline{I} & -\underline{R}_\infty \\ -\underline{R}_\infty & \underline{I} \end{bmatrix} \begin{bmatrix} \underline{E}^{+} e^{\underline{\kappa}(y-x)}(\underline{E}^{+})^{-1} & \underline{0} \\ \underline{0} & \underline{E}^{+} e^{-\underline{\kappa}(y-x)}(\underline{E}^{+})^{-1} \end{bmatrix} \begin{bmatrix} [\underline{I}-\underline{R}_\infty^2]^{-1} & \underline{0} \\ \underline{0} & [\underline{I}-\underline{R}_\infty^2]^{-1} \end{bmatrix} \begin{bmatrix} \underline{I} & \underline{R}_\infty \\ \underline{R}_\infty & \underline{I} \end{bmatrix} \tag{11.19}$$

Therefore $\underline{M}(x,y)$ is reducible to factors involving only the q×q matrices $\underline{\kappa}$, $\underline{E}^{+}$ and $\underline{R}_\infty$.

12. SURFACE BOUNDARY CONDITION: AUGMENTED MATRIX FORM

The preceding discussion has shown that the solution (9.23) of (6.5) literally cannot leave the air-water surface at level x unless we know the initial amplitude vector $\underline{A}(x)$ there. Our next main goal is to compute $\underline{A}(x)$. This requires attention to be redirected toward the air-water surface boundary conditions (5.18) and (5.19). These conditions show that the ℓ^{th} mode amplitudes $A_p^+(a;\ell)$ and $A_p^-(x;\ell)$ just above and below the surface are coupled to all other κ^{th} modes by virtue of their interaction with the directionally anisotropic surface. *Hence, to proceed, we must now reinstate the presence of the* p *and* ℓ *indexes in the notation.* Since we must consider all the amplitude nodes simultaneously, we shall form a vector from them, as shown below.

Now recall that the number q of components of the $1\times q$ vector $\underline{A}_p(y;\ell)$ depends on p and ℓ (cf. (5.10)-(5.15)): q is either m-1 or m, as the case may be. For the present boundary condition calculations we can treat all these special cases in a uniform manner by defining an *augmented amplitude vector* $\underline{A}_p(y;\ell)$ of m components for each $\ell = 0,\ldots,n$, regardless of whether p is 1 or 2:

$$\underline{A}_p^{\pm}(y;\ell) \equiv [A_p^{\pm}(y;1;\ell),\ldots,A_p^{\pm}(y;m;\ell)] \tag{12.1}$$

$$p = 1,2;\ \ell = 0,\ldots,n$$

The m^{th} components of these augmented amplitude vectors are either zero or $N^{\pm}(y;m,\cdot)$, in accordance with (5.6) and (5.7).

We next collect these n+1 augmented m-component vectors into one grand m(n+1) component vector for each flow:

$$\underline{A}_p^{\pm}(y) \equiv [\underline{A}_p^{\pm}(y;0),\ldots,\underline{A}_p^{\pm}(y;n)] \tag{12.2}$$

$p = 1,2.$

The m×m matrices $\hat{\underline{t}}_p(a,x;k|\ell)$, are now gathered up into one grand $m(n+1) \times m(n+1)$ matrix of the form $\hat{\underline{t}}_p(a,x)$, where we write

$$\hat{\underline{t}}_p(a,x) \equiv \begin{bmatrix} \hat{\underline{t}}_p(a,x;0|0) & \cdots & \hat{\underline{t}}_p(a,x;0|n) \\ \vdots & & \vdots \\ \hat{\underline{t}}_p(a,x;n|0) & \cdots & \hat{\underline{t}}_p(a,x;n|n) \end{bmatrix} \tag{12.3}$$

where p = 1 or 2. The remaining three $m(n+1) \times m(n+1)$ surface transfer matrices are constructed similarly. With these constructions (5.18) and (5.19) become

$$\underline{A}_p^{+}(a) = \underline{A}_p^{+}(x)\ \hat{\underline{t}}_p(x,a) + \underline{A}_p^{-}(a)\ \hat{\underline{r}}_p(a,x) \tag{12.4}$$

$$\underline{A}_p^{-}(x) = \underline{A}_p^{+}(x)\ \hat{\underline{r}}_p(x,a) + \underline{A}_p^{-}(a)\ \hat{\underline{t}}_p(a,x) \tag{12.5}$$

13. BOTTOM BOUNDARY CONDITION; AUGMENTED MATRIX FORM

Using the augmented m-component amplitude vectors $\underline{A}_p^{\pm}(y;\ell)$ defined in (12.1), we can reformulate bottom boundary condition (5.22) in the form

$$\underline{A}_p^{+}(z) = \underline{A}_p^{-}(z)\, \underline{\hat{r}}_p(z,b) \tag{13.1}$$

$$p = 1,2$$

where $\underline{A}_p^{\pm}(z)$ is now $1 \times m(n+1)$ and $\underline{\hat{r}}_p(z,b)$ is a $m(n+1) \times m(n+1)$ block diagonal matrix of the form

$$\underline{\hat{r}}_p(z,b) = \begin{bmatrix} \underline{\hat{r}}_p(z,b;0) & & \underline{0}_m \\ & \ddots & \\ \underline{0}_m & & \underline{\hat{r}}_p(z,b;n) \end{bmatrix} \tag{13.2}$$

for $p = 1,2$.

14. IMBED RULES

The imbed rules give the operators that yield the amplitudes at some internal level y of a layer X[x,z] knowing the $\underline{R}$ and $\underline{T}$ matrices for the two sublayers X[x,y] and X[y,z] above and below the level y, $x \le y \le z$ (cf. H.O., Vol. II, p. 297). In the present application of the imbed rule we are interested in finding the amplitudes $\underline{A}^{\pm}(x)$ just below the air-water surface in X[a,b], $a \le x \le y \le z \le b$, where X[a,x] is the upper surface boundary, X[x,y] is the water body, and X[z,b] is the lower boundary (a matte surface or a half space). Given the incident amplitude $\underline{A}_p^-(a) = [\underline{A}_p^-(a;0),\ldots,\underline{A}_p^-(a;n)]$, the required amplitudes $A^{\pm}(x) = [A^{\pm}(x;0),\ldots,A^{\pm}(x;n)]$ are given by

$$\underline{A}_p^-(x) = \underline{A}_p^-(a)\ \underline{\mathcal{T}}_p(a,x,b) \qquad 1 \times m(n+1) \tag{14.1}$$

$$\begin{aligned}\underline{A}_p^+(x) &= \underline{A}_p^-(x)\ \underline{R}_p(x,b) \qquad 1 \times m(n+1) \\ &= \underline{A}_p^-(a)\ \mathcal{R}_p(a,x,b)\end{aligned} \tag{14.2}$$

where the complete transmittance and complete reflectance operators are given by

$$\underline{\mathcal{T}}_p(a,x,b) = \underline{T}_p(a,x)\ [\underline{I} - \underline{R}_p(x,b)\ \underline{R}_p(x,a)]^{-1} \tag{14.3}$$

$$\underline{\mathcal{R}}_p(a,x,b) = \underline{\mathcal{T}}_p(a,x,b)\ \underline{R}_p(x,b) \tag{14.4}$$

These follow from the boundary condition (12.5) and the global interaction principles of §10 written for X[a,b] = X[a,x] U X[x,b]. In particular $T_p(a,x)$ takes the form $\hat{\underline{t}}_p(a,x)$, and $\underline{R}_p(x,a)$ takes the form $\hat{\underline{r}}_p(x,a)$, both of which are the $m(n+1) \times m(n+1)$ matrices defined in §13, above. The matrix $\underline{R}_p(x,b)$ is discussed in §15, below. Observe that by the isotropy of X[x,z] and X[z,b], (14.2) may be uncoupled and written as $n+1$ separate matrix equations for $1\times q$ vectors and $q\times q$ matrices, $q = m-1$ or m.

15. UNION RULES

The union rules give the $\underline{R}$ and $\underline{T}$ matrices of the union X(x,b) of two layers X(x,z) and X(z,b) (cf. H.O., Vol. IV, p. 30), knowing the $\underline{R}$ and $\underline{T}$ matrices of the two layers. In the present case X(x,z) is the water body and X(z,b) is its lower boundary. The required rules follow at once from bottom boundary condition (13.1) and the global interaction principles of §10 written for X[x,b] = X[x,z] U X[z,b]. They are

$$\underline{R}_p(x,b) = \underline{R}_p(x,z) + \underline{\mathcal{R}}_p(x,z,b)\ \underline{T}_p(z,x) \tag{15.1}$$

$$\underline{\mathcal{R}}_p(x,z,b) = \underline{T}_p(x,z)\ [I - \underline{R}_p(z,b)\ \underline{R}_p(z,x)]^{-1}\ \underline{R}_p(z,b) \tag{15.2}$$

In particular $\underline{R}_p(z,b)$ takes the form of the $m(n+1) \times m(n+1)$ augmented block diagonal matrix $\underline{\hat{r}}_p(z,b)$ defined in (13.2), while the four matrices $\underline{R}_p(x,z)$, $\underline{T}_p(z,x)$ and $\underline{R}_p(z,x)$, $\underline{T}_p(x,z)$ are $m(n+1) \times m(n+1)$ block diagonal matrices (augmented from $(m-1) \times (m-1)$ form by adding zeros in the m^{th} row and m^{th} columns, if necessary) made up of the ℓ-mode matrices associated with the water body. For example we have

$$\underline{R}_p(x,z) = \begin{bmatrix} \underline{R}_p(x,z;0) & \cdots & \underline{0}_m \\ \vdots & \ddots & \vdots \\ \underline{0}_m & \cdots & \underline{R}_p(x,z;n) \end{bmatrix} \tag{15.3}$$

The $m \times m$ matrix $\underline{R}_p(x,z;\ell)$, $\ell = 0,\ldots,n$, is given by (10.9), and, as noted, is augmented to $m \times m$, if necessary, for use in (15.3) with the augmented $1 \times m$ amplitude vectors in (12.1). The remaining three augmented matrices of the water body are assembled into block diagonal form similarly.

Another application of the union rule, this time to the union of X[a,x] (the upper surface) and X[x,b] (the water body plus the lower boundary) yields the matrix needed to find the upward radiance amplitudes $\underline{A}_p^+(a)$ emerging from the air-water surface of the hydrosol. The required rules follow from boundary conditions (12.4) and (12.5) and the global interaction principles of §10 written for $X[a,b] = X[a,x] \cup X[x,b]$. The resultant union rules are

$$\underline{R}_p(a,b) = \underline{R}_p(a,x) + \underline{\mathcal{R}}_p(a,x,b)\ \underline{T}_p(x,a) \tag{15.4}$$

$$\underline{\mathcal{R}}_p(a,x,b) = \underline{T}_p(a,x)\ [\underline{I} - \underline{R}_p(x,b)\ \underline{R}_p(x,a)]^{-1}\ \underline{R}_p(x,b) \tag{15.5}$$

Here $\underline{R}_p(a,x)$, $\underline{T}_p(x,a)$ and $\underline{R}_p(x,a)$, $\underline{T}_p(a,x)$ are the four $m(n+1) \times m(n+1)$ transfer matrices of the upper surface occurring in (12.4) and (12.5), while $\underline{R}_p(x,b)$ is the matrix found in (15.1).

The required upward emergent radiance amplitude $\underline{A}_p^+(a)$ leaving X[a,b] are given by

$$\underline{A}_p^+(a) = \underline{A}_p^-(a)\ \underline{R}_p(a,b) \qquad 1 \times m(n+1) \tag{15.6}$$

16. SOLUTION FOR A FINITELY DEEP MEDIUM

Having determined $\underline{A}_p^{\pm}(x)$ via (14.1) and (14.2), we may now return to (9.23) and find numerical values of $A^{\pm}(y;u)$ (with p and ℓ understood) at all depths y in the homogeneous water body part X[x,z] of the complete medium X[a,b] = X[a,x] U X[x,z] U X[z,b]. Indeed, we may now explicitly evaluate the initial amplitudes $a_j^{\pm}(x)$ in terms of the two basic q×q reflectance matrices $\underline{R}(x,b)$ and $\underline{R}_\infty$ of the medium X[a,b]. First observe that from (11.4) we have

$$\underline{e}_j^- = -\underline{R}_\infty\, \underline{e}_j^+ \quad , \qquad j = 1,\ldots,q \tag{16.1}$$

This, with the mode-uncoupled form of (14.2), allows us to rewrite (9.19) as

$$\begin{aligned} a_j^+(x) &= \underline{A}^+(x)\, \underline{e}_j^+ + \underline{A}^-(x)\, \underline{e}_j^- \\ &= \underline{A}^-(x)\, \underline{R}(x,b)\, \underline{e}_j^+ - \underline{A}^-(x)\, \underline{R}_\infty\, \underline{e}_j^+ \\ &= \underline{A}^-(x)\, [\underline{R}(x,b) - \underline{R}_\infty]\underline{e}_j^+ \end{aligned} \tag{16.2}$$

Moreover, (9.20) becomes

$$\begin{aligned} a_j^-(x) &= -\underline{A}^-(x)\, \underline{R}(x,b)\, \underline{R}_\infty\, \underline{e}_j^+ + \underline{A}^-(x)\, e_j^+ \\ &= \underline{A}^-(x)\, [\underline{I} - \underline{R}(x,b)\, \underline{R}_\infty]\underline{e}_j^+ \end{aligned} \tag{16.3}$$

where r and ℓ are understood in (16.2) and (16.3). Hence if the optical properties of the medium are known, $\underline{A}^-(x)$ ($\equiv \underline{A}_p^-(x)$ obtained via (14.1)) will be the only additional piece of information needed for a full solution.

17. SOLUTION FOR AN INFINITELY DEEP MEDIUM

On letting $z \to \infty$ in (16.2) and (16.3) (so that also $b \to \infty$) we can evaluate the amplitude coefficients $a_j^{\pm}(x)$ in (9.23) for the case of an infinitely deep homogeneous medium $X[x,\infty]$, $a \le x < z \le b = \infty$. Noting that z appears implicitly in (16.2) and (16.3) via $\underline{R}(x,b)$ (cf. (15.1)) and recalling (11.1), we find, for the limiting case $z \to \infty$, that

$$a_j^{+}(x) = 0 \quad , \quad j = 1,\ldots,q \tag{17.1}$$

and

$$a_j^{-}(x) = \underline{A}^{-}(x)\,[\underline{I} - \underline{R}_{\infty}^{2}]\,\underline{e}_j^{+} \quad , \quad j = 1,\ldots,q \tag{17.2}$$

Thus (9.23) reduces in this case to the amplitudes with purely decaying modes:

$$\boxed{A^{\pm}(y;u) = \sum_{j=1}^{q} a_j^{-}(x)\, e^{-\kappa_j (y-x)}\, f_j^{\mp}(u)} \tag{17.3}$$

where p and ℓ are understood.

18. THE ASYMPTOTIC RADIANCE DISTRIBUTION

When y in (17.3) becomes large, the associated directional distribution of the zero mode cosine radiance amplitudes takes a well-defined form, that of the so-called *asymptotic radiance distribution*. The basis for this is the following. Recall that we have arranged the distinct, positive eigenvalues of each mode ℓ in ascending order: $0 < \kappa_1(\ell) < \kappa_2(\ell) < \cdots < \kappa_q(\ell)$.

Numerical experiments with realistic models (cf. §8D) invariably yield the inequalities:

$$\kappa_1(0) < \kappa_1(\ell) \text{ for all } \ell=1,\ldots,n. \tag{18.1}$$

Physically, this is interpreted as showing that locally high curvature and asymmetry of the radiance distribution (thought of as a surface in three-dimensional space) tends to be smoothed and reduced as depth y increases. An intuitive theoretical proof of (18.1) can be given along the lines developed in Preisendorfer (1959). Now let us multiply each side of (17.3) for the case $p = 1$, $\ell = 0$, by $e^{\kappa_1(0)(y-x)}$ and then take the limit:

$$\boxed{\begin{array}{l} A^{\pm}(\infty;u) \equiv \lim\limits_{y\to\infty} A_1^{\pm}(y;u;0)\ e^{\kappa_1(0)(y-x)} = a_1^{-}(x;0)\ f_1^{\mp}(u;0) \\ u = 1,\ldots,m \end{array}} \tag{18.2}$$

We call the set of numbers $\{A^{\pm}(\infty;1),\ldots,A^{\pm}(\infty;m)\}$ or any scalar multiple of this set the *asymptotic radiance distribution* (of order m). Thus $A^{\pm}(\infty;u)$ is defined via the zero mode ($\ell=0$) cosine radiance amplitude ($p = 1$) of the radiance field. All other modes of the physical radiance distribution decay at a greater depth rate than $\kappa_1(0)$, by (18.1), and are lost on the way to $y = \infty$. Hence the asymptotic radiance distribution is symmetrical about the vertical direction. The u-dependence of $f_1^{\pm}(u;0)$ defines the zenith to nadir shape of the asymptotic radiance distribution. As we have seen, $f_1^{\pm}(u;0)$ is determined solely by the m×m system matrix $\underline{K}(0)$ (for the zeroth azimuthal mode), which in turn is defined by the shape of the volume scattering function. Observe that all directional information about the initial radiance distribution via $\underline{A}_1^{\pm}(x;0)$ has been lost in the formation of $a_1^{-}(x;0)$. Hence the asymptotic radiance distribution is an inherent optical property of an infinitely deep homogeneous medium.

19. TWO-FLOW IRRADIANCE OBSERVABLES ASSOCIATED WITH THE ASYMPTOTIC RADIANCE DISTRIBUTION

The three main two-flow irradiance observables, namely the $K(y,\pm)$, $R(y,\pm)$, and $D(y,\pm)$ quantities (H.O., Vol. V, p. 115-118), have specific values in an asymptotic radiance field. Observe that these quantities will be independent of y, and henceforth we shall denote their constant values, respectively, by "$K_\pm$", "$R_\pm$", and "$D_\pm$".

First, at great depths $K_+ = K_- \equiv k_\infty$ and the common value k_∞ is related to $\kappa_1(0)$ by

$$\kappa_1(0) = k_\infty/\alpha \tag{19.1}$$

Hence, knowing $\kappa_1(0)$ and α, we can deduce the asymptotic decay value (cf. H.O., Vol. V, pp. 244-248):

$$\boxed{k_\infty = \alpha\kappa_1(0) \qquad (m^{-1})} \tag{19.2}$$

common to *all* radiometric magnitudes (scalar irradiance, radiance, etc.). Further, we find

$$R_\infty \equiv R_+^{-1} = R_- = \frac{\sum_{u=1}^{m} A^+(\infty;u)|\mu_u|\Delta\mu}{\sum_{u=1}^{m} A^-(\infty;u)|\mu_u|\Delta\mu} = \frac{\sum_{u=1}^{m} f_1^-(u;0)|\mu_u|}{\sum_{u=1}^{m} f_1^+(u;0)|\mu_u|} \tag{19.3}$$

Finally

$$D_\pm = \frac{\sum_{u=1}^{m} f_1^\mp(u;0)}{\sum_{u=1}^{m} f^\mp(u;0)|\mu_u|} \tag{19.4}$$

From these one finds the volume absorption coefficient (cf. H.O., Vol. V, p. 247):

$$a = \frac{k_\infty(1-R_\infty)}{D_- - R_\infty D_+} \tag{19.5}$$

which can serve as a check on the computations. Recall that the whole problem began with α and σ given, so that $a = \alpha - s$, derived from the initially given data, can be checked against a in (19.5), the end of a long chain of arithmetic operations.

Interestingly, from (19.1)-(19.4) we see that $K(y,\pm)$, $R(y,\pm)$, and $D(y,\pm)$, which are apparent optical properties for small depths y, attain the status of inherent optical properties in the limit of arbitrarily great depth.

20. INVERSE SOLUTION

In the present homogeneous setting we can determine the local reflectance and transmittance matrices $\underline{\hat{\rho}}$ and $\underline{\hat{\tau}}$ for each azimuthal mode from observations of the radiance distribution at various depths. We shall now give some attention to this matter.

First of all observe that, by knowing $\underline{\hat{\rho}}$ and $\underline{\hat{\tau}}$ from the inverse solution, we can in effect estimate the quad-averaged volume scattering function values $sp(u',v'|u,v)$ for all distinct quad pairs $Q_{u'v'}$ and Q_{uv} in an adopted decomposition of the unit sphere. Also estimable is $\alpha' = \alpha - sp(u,v|u,v)$, the observable value of the volume attenuation coefficient α (for a discussion of this interesting phenomenon of the inaccessibility of α and hence $p(a,v|u,v)$, see H.O., Vol. VI, p. 296). The smaller the solid angle subtense of Q_{uv}, the closer will α' be to the "true" value α. Hence a finite sequence of ever finer Q_{uv} partitions will allow α to be estimated as a limited value. Since the $p(u',v'|u,v)$ must add up to 1 as we sum over all Q_{uv} for any fixed $Q_{u'v'}$, we can then estimate also s and $p(u,v|u,v)$ from the inversion results $\underline{\hat{\rho}}$ and $\underline{\hat{\tau}}$. This then amounts to a practical recovery of α and σ to within the degree of accuracy governed by the fineness of the quad decomposition of the unit sphere.

To start the inverse procedure, we obtain the radiance amplitude vectors $\underline{A}(y)$ at various depths (to be specified below) from the given, observed radiance distributions at those depths.

The following steps in the inverse procedure are based on some observations in Preisendorfer (1968) which may be readily applied now that we have the specific solution for $\underline{A}(y)$ available in the eigenmatrix form:

$$\underline{A}(y) = A(x)\,\underline{M}(x,y) = \underline{A}(x)\,\underline{M}(y-x) \tag{20.1}$$

where

$$\underline{M}(y-x) \equiv \underline{E}\,\exp[\underline{\kappa}(y-x)]\,\underline{E}^{-1}. \tag{20.2}$$

Since $\underline{A}(y) = [A^{+}(y), \underline{A}^{-}(y)]$ and $\underline{A}^{\pm}(y)$ are q-dimensional vectors, to invert (20.1) we need 2q linearly independent observed amplitude vectors $\underline{A}(y_j)$, $j = 1,..,2q$ of the light field. These observations are spaced at depths some constant interval $\Delta y = y_{j+1}-y_j$ apart. Then by the mapping property (6.10), for each $j = 1,\ldots,q$, we have

$$\begin{aligned}\underline{A}(y_{j+1}) &= \underline{A}(y_j)\,\underline{E}\,\exp[\underline{\kappa}\Delta y]\,\underline{E}^{-1}\\ &\equiv \underline{A}(y_j)\,\underline{M}(\Delta y)\,.\end{aligned} \tag{20.3}$$

That the set of vectors $\underline{A}(y_j)$, $j = 1,\ldots,2q$ is in principle linearly independent follows from the fact that the set of exponential functions $\exp[\pm\kappa_j(y-x)]$, $j = 1,\ldots,q$, is linearly independent (recall the discussion leading to (7.11)).

Next, write

$$\underline{A}_1 \equiv \begin{bmatrix}\underline{A}(y_1)\\ \vdots\\ \underline{A}(y_{2q})\end{bmatrix} \qquad (2q \times 2q) \tag{20.4}$$

From what we have just observed $\underline{A}_1$ is an invertible $2q \times 2q$ matrix. Furthermore write

$$\underline{A}_2 \equiv \begin{bmatrix}\underline{A}(y_2)\\ \vdots\\ \underline{A}(y_{2q+1})\end{bmatrix} \qquad (2q \times 2q) \tag{20.5}$$

Then the set of relations (20.3) can be written

$$\underline{A}_2 = \underline{A}_1\ \underline{M}(\Delta y) \tag{20.6}$$

whence

$$\underline{M}(\Delta y) = \underline{A}_1^{-1}\underline{A}_2 \tag{20.7}$$

We are next led to find the eigenvector matrix $\underline{E}$ and eigenvalue matrix $\underline{\kappa} = \mathrm{diag}[\kappa_1,\ldots,\kappa_q, -\kappa_1,\ldots,-\kappa_q]$, as follows.

Evidently by inspection of (20.2), the eigenvectors of $\underline{M}(y-x)$ are already $\underline{E}^{\pm} = [\underline{e}_1^{\pm},\ldots,\underline{e}_q^{\pm}]$, the eigenvectors of $\underline{K}$. Next, suppose we order and then label the 2q eigenvalues of $\underline{M}(\Delta y)$ as follows:

$$\gamma_q^- < \gamma_{q-1}^- < \cdots < \gamma_2^- < \gamma_1^- < 1 < \gamma_1^+ < \gamma_2^+ < \cdots < \gamma_{q-1}^+ < \gamma_q^+ \tag{20.8}$$

By (20.2) we expect that

$$\gamma_j^{\pm} = \exp[\pm\kappa_j\Delta y] \quad , \quad j = 1,\ldots,q \tag{20.9}$$

Therefore, we write

$$\kappa_j^{\pm} \equiv \pm(\Delta y)^{-1}\ \ell n\ \gamma_j^{\pm} \quad , \quad j = 1,\ldots,q \tag{20.10}$$

Of course when working with actual data we will find that we don't exactly have $\gamma_j^+\gamma_j^- = 1$, with the result that we will not exactly have $\kappa_j^+ = -\kappa_j^-$. We will then adjust the $\kappa_j^{\pm}$ found above so that we do obtain κ_j with this symmetry property. We accordingly set

$$\kappa_j \equiv \tfrac{1}{2}(\kappa_j^+ - \kappa_j^-) \qquad j = 1,\ldots,q \tag{20.11}$$

The $\underline{E}$ are re-computed for these symmetrized κ_j. Once $\underline{E}$ and the κ_j, $j = 1,\ldots,q$ have been so found, we can then go on to compute

$$\underline{K} = \underline{E}\ \underline{\kappa}\ \underline{E}^{-1} \qquad (2q \times 2q) \tag{20.12}$$

which yields the two estimated q×q matrices $\hat{\underline{\rho}}$, $\hat{\underline{\tau}}$, discussed above, for each mode $\ell = 0,\ldots,n$; whence α and σ.

As a check on the preceding procedure we can construct the amplitude vectors $\underline{A}(y_j)$, $j = 1,\ldots,2q$ from the estimated $\underline{K}$ in (20.12) and compare with the observed values. Note that the eigenstructure $\kappa_j^{\pm}$ in (20.10) and the $\underline{E}^{\pm}$ of $\underline{M}(\Delta y)$ in (20.7) (before symmetrizing in (20.11)) will reproduce the observed radiance amplitudes $\underline{A}(y_j)$ in (20.5) exactly. Once the symmetrized κ_j and their associated $\underline{E}^{\pm}$ matrices have been found, we will have in effect estimated the system matrix $\underline{K}$ of the homogeneous medium, i.e., we will know the inherent optical properties α and σ of the medium. Let $\hat{\underline{K}}$ be the estimate. When new incident light fields $\underline{A}'(x)$ come along, the associated new $\underline{A}(y)$, say $\underline{A}'(y)$, at any depth y (and hence $N(y;u,v)$ at the depth y), can be obtained from the mapping property $\underline{A}'(y) = \underline{A}'(x)\exp[\hat{\underline{K}}(y-x)]$.

APPENDIX A

EIGENMATRIX THEORY OF THE TWO-FLOW IRRADIANCE MODEL

1. *Introduction*

The eigenmatrix theory of §7 has a simpler counterpart in the form of the eigenanalysis of the two-flow irradiance model. We shall develop this simpler version here, following the main developments of §7 along a parallel track, as far as possible. It will be an instructive exercise on two counts: First, the key q×q matrices $\underline{E}^{\pm}$ and $\underline{F}^{\pm}$ will reduce to numbers (because now q = 1) and so we will be able to see their physical constitution directly; all formulas can, if desired, be numerically evaluated by hand, and simple algebraic operations reveal all the inner workings of the eigenmatrix theory. Hence the present discussion can serve as an informative prerequisite to the main study of §7. Second, the present model, despite its simplicity, is actually a bit more complex in the sense that the local optical properties $\tau_{\pm}$ and $\rho_{\pm}$, which are the present counterparts to the q×q matrices $\underline{\hat{\tau}}$, $\underline{\hat{\rho}}$, are in fact anisotropic; that is, unlike the radiance case of §7, we must distinguish between absorption and backscattering activities on the upward and downward flows on the local level.

2. *The Two-Flow Irradiance Equations*

The form of the irradiance model we shall use is that developed in Preisendorfer and Mobley (1984). At any *geometric depth* y, x ≤ y ≤ z,

$$\mp \frac{d}{dy} H(y,\pm) = \tau_{\pm} H(y,\pm) + \rho_{\mp} H(y,\mp) \tag{A2.1}$$

where $\tau_{\pm} = -[a_{\pm} + \underline{b}_{\pm}]$

and

$$a_{\pm} = D_{\pm} a$$
$$b_{\pm} = D_{\pm} \bar{b}$$
$$\rho_{\pm} = b_{\pm}$$

We shall use the $\rho_{\pm}$ and $b_{\pm}$ notation interchangeably, according to the momentary interpretation desired: local reflectance or local backscatter. This model has four depth-independent parameters: the two distribution factors $D_{\pm}$, which describe the mean path length of ascending (+) and descending (-) photons through a layer of unit thickness; and two inherent optical properties, the volume absorption coefficient a and the mean backscatter coefficient $\bar{b}$. We shall work with *realistic media* (cf. §8D), i.e., those for which $a > 0$, $b > 0$ and $D_{+} > D_{-} > 0$.

The backscatter coefficient $\bar{b}$ is an inherent property in the following sense. By definition, the general depth-dependent backscatter function $b(y,\pm)$ is

$$b(y,\pm) = H^{-1}(y,\pm) \int_{\Xi_{\mp}} d\Omega(\xi) \int_{\Xi_{\pm}} N(y;\xi')\, \sigma(y;\xi';\xi)\, d\Omega(\xi') \qquad \text{(A2.2)}$$

In the two-flow irradiance model we may adopt an average radiance over $\Xi_{\pm}$ when evaluating (A2.2). This radiance is a form of quad-averaged radiance, with the quad replaced by Ξ_{+} or Ξ_{-}. Thus in (A2.2) we may adopt radiances over $\Xi_{\pm}$ in the form

$$N(y,\xi) = \begin{cases} h(y,+)/2\pi & \text{if} \quad \xi \text{ is in } \Xi_{+} \\ h(y,-)/2\pi & \text{if} \quad \xi \text{ is in } \Xi_{-} \end{cases} \qquad \text{(A2.3)}$$

Under this hypothesis, (A2.2) reduces to

$$b(y,\pm) = D(y,\pm)\ \bar{b}(y) \tag{A2.4}$$

where we have written

$$\bar{b}(y) \equiv \frac{1}{2\pi} \int_{\Xi_-} d\Omega(\xi) \int_{\Xi_+} \sigma(y;\xi';\xi)\ d\Omega(\xi') \tag{A2.5}$$

$$\text{or} \quad \frac{1}{2\pi} \int_{\Xi_+} d\Omega(\xi) \int_{\Xi_-} \sigma(y;\xi';\xi)\ d\Omega(\xi')$$

The isotropy of σ at depth y implies that $\sigma(y;\xi';\xi) = \sigma(y;\xi;\xi')$ for all ξ' and ξ. Hence $\bar{b}(y)$ does not depend on the direction of the incident flow of photons across the plane at level y, and so the alternate form of $\bar{b}(y)$ in (A2.5) also characterizes $\bar{b}(y)$. In either of the two formulas, observe that we are finding a mean or average magnitude of the *backward* scatter of photons across the horizontal plane at level y, $x \le y \le z$. Since $\bar{b}(y)$ in this sense is independent of the direction of flow of the photons, it is an inherent optical property under the hypothesis (A2.3). Note that when scattering is spherically symmetric, i.e., when $\sigma(y;\xi';\xi) = s(y)/4\pi$, then $\bar{b}(y) = s(y)/2$, i.e., $\bar{b}(y)$ is half the volume total scattering function s(y), as expected.

3. *The Fundamental Solution of the Two-Flow Model*

We may write (A2.1) in matrix form as

$$\frac{d}{dy}\ \underline{H}(y) = \underline{H}(y)\ \underline{K} \tag{A3.1}$$

where we have written

$$\underline{H}(y) \equiv [H(y,+),\ H(y,-)] \qquad (1\times 2) \qquad \text{(A3.2)}$$

and

$$\underline{K} \equiv \begin{bmatrix} -\tau_+ & \rho_+ \\ -\rho_- & \tau_- \end{bmatrix} \qquad (2\times 2) \qquad \text{(A3.3)}$$

In this way we see (A3.1) as the irradiance counterpart to (6.5). Notice that we now have τ dependent on the upward (+) or downward (-) flow. Hence we do not have the local isotropy that is present in (6.4).

Following the development in §6, we can write the fundamental solution of (A3.1) as

$$\underline{M}(x,y) = \exp[\underline{K}(y-x)] \qquad \text{(A3.4)}$$

where $\underline{K}$ is now the relatively simple 2×2 matrix in (A3.3). It has all the properties of the $2q \times 2q$ matrix $\underline{M}(x,y)$ of §6. In particular, the mapping property holds:

$$\underline{H}(y) = \underline{H}(x)\ \underline{M}(x,y) \qquad \text{(A3.5)}$$

We are in effect working with $\underline{M}(x,y)$ of §6 for the case $q = 1$. Keep in mind, however, that the parallelism between the multimode theory and the present irradiance theory is not exact, since (A2.1) exhibits anisotropy via the property $D_+ > D_- > 0$.

4. *The Eigentheory of Two-Flow Irradiance Fields*

The purely exponential basis functions $B_j^{\pm}(y)$ of (7.1) have their present counterparts in two scalar-valued functions $B(y,\pm)$, since we are in effect working in the $q = 1$ case. Thus we postulate for the irradiance model two exponentially varying functions $B(y,\pm)$ such that

$$B(y,\pm) = B(x,\pm)\exp[k_{\pm}(y-x)] \qquad [\mathrm{W\cdot m^{-2}}] \qquad x \le y \le z \tag{A4.1}$$

For reasons which will become clear shortly, $B(y,\pm)$ are the *eigen-irradiances* of the medium. Linear combinations of these are to represent the observable irradiance fields

$$\begin{aligned} H(y,+) &= B(y,+)\, f_{++} + B(y,-)\, f_{-+} \\ H(y,-) &= B(y,+)\, f_{+-} + B(y,-)\, f_{--} \end{aligned} \tag{A4.2}$$

where $f_{\pm\pm}$ and $f_{\mp\pm}$ are dimensionless constants. These equations may be placed in matrix form:

$$\underline{H}(y) = \underline{B}(y)\,\underline{F} \tag{A4.3}$$

where we have written

$$\underline{B}(y) \equiv [B(y,+),\, B(y,-)] \qquad (1\times 2) \tag{A4.4}$$

$$\underline{F} \equiv \begin{bmatrix} f_{++} & f_{+-} \\ f_{-+} & f_{--} \end{bmatrix} \qquad (2\times 2) \tag{A4.5}$$

Conversely, we can represent the eigen-irradiances $B(y,\pm)$ as linear combinations of the observable irradiances $H(y,\pm)$:

$$B(y,+) = H(y,+)\ e_{++} + H(y,-)\ e_{-+} \tag{A4.6}$$
$$B(y,-) = H(y,+)\ e_{+-} + H(y,-)\ e_{--}$$

where $e_{\pm\pm}$, $e_{\pm\mp}$ are dimensionless constants. These equations may be written more compactly as

$$\underline{B}(y) = \underline{H}(y)\ \underline{E} \tag{A4.7}$$

where we have written

$$\underline{E} \equiv \begin{bmatrix} e_{++} & e_{+-} \\ e_{-+} & e_{--} \end{bmatrix} \tag{A4.8}$$

Clearly

$$\underline{F} = \underline{E}^{-1} \tag{A4.9}$$

analogous to (7.24).

Corresponding to the law of change of $\underline{H}(y)$ in (A3.1), that for $\underline{B}(y)$ is, by (A4.1),

$$\frac{d}{dy}\ \underline{B}(y) = \underline{B}(y)\ \underline{k} \tag{A4.10}$$

where

$$\underline{k} = \text{diag}[k_+, k_-] \qquad (2\times2)$$

Observe that $k_{\pm}$ have units m^{-1} because y is now geometric depth, rather than optical depth as used in the body of this report. Geometric depth is adopted here as the more natural depth, since the irradiance model does not have the volume attenuation coefficient α to readily convert geometric depth to optical depth. (If it is still desired to work with y as optical depth in (A2.1) and (A4.10), then $a_{\pm}$, $b_{\pm}$ and $k_{\pm}$ must be divided by α.)

Following the procedure leading to (7.27) we now find, via (A3.1) and (A4.10), in the present case that

$$\underline{K}\,\underline{E} = \underline{E}\,\underline{k} \;, \tag{A4.11}$$

which is the basic eigenstructure equation for the irradiance model.

Equation (A4.11) contains two eigenvector/eigenvalue statements. The first may be written

$$\begin{bmatrix} -\tau_+ & \rho_+ \\ -\rho_- & \tau_- \end{bmatrix} \begin{bmatrix} e_{++} \\ e_{-+} \end{bmatrix} = k_+ \begin{bmatrix} e_{++} \\ e_{-+} \end{bmatrix} \tag{A4.12}$$

In component form this becomes

$$\begin{aligned} -\tau_+ \, e_{++} + \rho_+ \, e_{-+} &= k_+ \, e_{++} \\ -\rho_- \, e_{++} + \tau_- \, e_{-+} &= k_+ \, e_{-+} \end{aligned} \tag{A4.13}$$

The two unknowns e_{++} and e_{-+} are determined by (A4.13) up to a common factor. The first equation of (A4.13) suggests that we can set

$$e_{++} \equiv c_1\rho_+ \quad , \quad e_{-+} \equiv c_1(\tau_+ + k_+) \tag{A4.14}$$

where $c_1 = 1$ m. Thus e_{++} has the magnitude of ρ_+ and is dimensionless.* The second of (A4.13), with these values of e_{++} and e_{-+} becomes

$$-(\tau_+ + k_+)(\tau_- - k_+) + \rho_+\rho_- = 0 \tag{A4.15}$$

We shall return to this equation in a moment.

The second eigenvector/eigenvalue equation in (A4.11) is

$$\begin{bmatrix} -\tau_+ & \rho_+ \\ -\rho_- & \tau_- \end{bmatrix} \begin{bmatrix} e_{+-} \\ e_{--} \end{bmatrix} = k_- \begin{bmatrix} e_{+-} \\ e_{--} \end{bmatrix} \tag{A4.16}$$

In component form this is

$$\begin{aligned} -\tau_+ e_{+-} + \rho_+ e_{--} &= k_- e_{+-} \\ -\rho_- e_{+-} + \tau_- e_{--} &= k_- e_{--} \end{aligned} \tag{A4.17}$$

The second of (A4.17) suggests that, on using the same c_1 as in (A4.14), we set

$$e_{+-} \equiv c_1(\tau_- - k_-) \quad , \quad e_{--} \equiv c_1\rho_- \tag{A4.18}$$

The first of (A4.17) yields (A4.15), but now with k_- instead of k_+. Hence the

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* In all subsequent uses of (A4.14) and (A4.18), c_1 will not explicitly appear, since its purpose is simply to make the $e_{\pm\pm}$, $e_{\pm\mp}$ dimensionless. If we had adopted optical depth y in (A2.1), the $\tau_\pm$ and $\rho_\pm$ would be replaced by the dimensionless quantities $\tau_\pm/\alpha$, $\rho_\pm/\alpha$ and c_1 would not be needed at this stage.

eigenvalues $k_{\pm}$ are governed by the quadratic equation

$$(k + \tau_+)(k - \tau_-) + \rho_+\rho_- = 0 \tag{A4.19}$$

or

$$k^2 + (\tau_+ - \tau_-)k + (\rho_+\rho_- - \tau_+\tau_-) = 0 \tag{A4.20}$$

From this we see that the roots $k_{\pm}$ of (A4.19) satisfy the relations

$$\begin{aligned} k_+ + k_- &= \tau_- - \tau_+ \\ &= [D_+ - D_-][a + \bar{b}] > 0 \end{aligned} \tag{A4.21}$$

$$\begin{aligned} k_+k_- &= \rho_+\rho_- - \tau_+\tau_- \\ &= -D_+D_-a[a + 2\bar{b}] < 0 \end{aligned} \tag{A4.22}$$

The roots themselves are given by

$$\begin{aligned} k_{\pm} &= \tfrac{1}{2}\{(\tau_- - \tau_+) \pm [(\tau_+ - \tau_-)^2 - 4(\rho_+\rho_- - \tau_+\tau_-)]^{\frac{1}{2}}\} \\ &= \tfrac{1}{2}\{(\tau_- - \tau_+) \pm [(\tau_+ + \tau_-)^2 - 4\rho_+\rho_-]^{\frac{1}{2}}\} \end{aligned} \tag{A4.23}$$

For realistic media ($a > 0$; see §8D) it follows from (A4.22) that k_+ and k_- are nonzero and of opposite signs. From this we conclude that

$$k_- < 0 < k_+ \tag{A4.24}$$

Under most natural lighting conditions (light from a sunny or overcast sky entering a lake or sea), $D_+ > D_-$, and so from (A4.21) we have $k_+ + k_- > 0$; in other words $|k_-| < k_+$. Therefore, according to this model, in natural waters downwelling eigen-irradiance usually decays slower than upwelling eigen-irradiance.

Moreover, if the medium exhibits back scattering, i.e., $b > 0$, then from (A4.23) and recalling that $\tau_\pm$ are negative, we observe that

$$\begin{aligned} \delta &\equiv \tau_+ + k_+ = \tau_- - k_- \\ &= \tfrac{1}{2}\{(\tau_+ + \tau_-) + [(\tau_+ + \tau_-)^2 - 4\rho_+\rho_-]^{\frac{1}{2}}\} < 0 \end{aligned} \qquad \text{(A4.25)}$$

These inequalities yield the following physical interpretations about decay rates of the photon streams:

$$k_+ < a_+ + b_+ \quad \text{and} \quad -k_- < a_- + b_- . \qquad \text{(A4.26)}$$

From (A4.19) we obtain what will turn out to be key ratios:

$$\frac{\rho_+}{\tau_+ + k_\pm} = \frac{\tau_- - k_\pm}{\rho_-} \qquad \text{(A4.27)}$$

and, after some rearranging, we find from this also that

$$\mp \left[\frac{(\rho_+\rho_-)k_\pm \; - \; \delta^2 k_\mp}{\rho_+\rho_- - \delta^2}\right] = \tau_\pm \qquad \text{(A4.28)}$$

The combination $\rho_+\rho_- - \delta^2$ in (A4.28) is never zero in realistic backscattering media; for we have by (A4.24) and (A4.25),

$$\Delta \equiv \rho_+\rho_- - \delta^2 = \delta(k_- - k_+) > 0 \tag{A4.29}$$

Unlike the isotropic setting of §7, the eigenvalues $k_\pm$ of (A4.11) are not of equal magnitude; although, by (A4.22), they are of opposite sign. By (A4.22) they will be of equal magnitude and we will have isotropy if $\tau_+ = \tau_-$, i.e., if in (A2.1) we have $D_+ = D_-$. This condition unfortunately is never satisfied by irradiance fields in realistic media; and so we should retain distinct values of D_+ and D_- in the present irradiance model. Occasionally, however, (cf. Preisendorfer and Mobley, 1984, or H.O., Vol. V, p. 64) it is of interest to consider the one-D case to explore potential symmetries, and develop very simple light field models.

We may summarize the preceding findings about the eigenmatrix $\underline{E}$ in the form

$$\underline{E} = \begin{bmatrix} e_{++} & e_{+-} \\ e_{-+} & e_{--} \end{bmatrix} = \begin{bmatrix} \rho_+ & \delta \\ \delta & \rho_- \end{bmatrix} \tag{A4.30}$$

The inverse $\underline{F}$ of $\underline{E}$ may therefore be represented as

$$\underline{F} = \begin{bmatrix} f_{++} & f_{+-} \\ f_{-+} & f_{--} \end{bmatrix} = \Delta^{-1} \begin{bmatrix} \rho_- & -\delta \\ -\delta & \rho_+ \end{bmatrix} \tag{A4.31}$$

5. *Eigenmatrix Representation of the Two-Flow Model Fundamental Solution*

We may now return to (A3.4) and write it in a form that is parallel to (7.29):

$$\underline{M}(x,y) = \underline{E}\,\exp[\underline{k}(y-x)]\underline{F} \qquad (2\times 2) \tag{A5.1}$$

where $\underline{E}$ and $\underline{F}$ are given in (A4.30) and (A4.31), and $\underline{k} = \mathrm{diag}[k_+,k_-]$, with $k_\pm$ as defined in (A4.23). In more detail, we have from (A5.1) the following scalar counterparts to (9.3)-(9.6):

$$M_{++}(x,y) = e_{++}\, e^{k_+(y-x)}\, f_{++} + e_{+-} e^{k_-(y-x)}\, f_{-+} \tag{A5.2}$$

$$M_{-+}(x,y) = e_{-+}\, e^{k_+(y-x)}\, f_{++} + e_{--} e^{k_-(y-x)}\, f_{-+} \tag{A5.3}$$

$$M_{+-}(x,y) = e_{++}\, e^{k_+(y-x)}\, f_{+-} + e_{+-} e^{k_-(y-x)}\, f_{--} \tag{A5.4}$$

$$M_{--}(x,y) = e_{-+}\, e^{k_+(y-x)}\, f_{+-} + e_{--} e^{k_-(y-x)}\, f_{--} \tag{A5.5}$$

We will occasionally use the $e_{++},\ldots,f_{--}$ notation instead of the $\rho_\pm,\delta$ notation since the former notation acts as a useful mnemonic. However, for reference, we also can write (A5.2)-(A5.5) in the form

$$M_{++}(x,y) = [\rho_+\rho_-\, e^{k_+(y-x)} - \delta^2\, e^{k_-(y-x)}]/\Delta \tag{A5.6}$$

$$M_{-+}(x,y) = \rho_-\delta[e^{k_+(y-x)} - e^{k_-(y-x)}]/\Delta \tag{A5.7}$$

$$M_{+-}(x,y) = -\rho_+\delta[e^{k_+(y-x)} - e^{k_-(y-x)}]/\Delta \tag{A5.8}$$

$$M_{--}(x,y) = -[\delta^2\, e^{k_+(y-x)} - \rho_+\rho_-\, e^{k_-(y-x)}]/\Delta \tag{A5.9}$$

It is at once clear that

$$\begin{aligned} M_{++}(x,x) &= M_{--}(x,x) = 1 \\ M_{+-}(x,x) &= M_{-+}(x,x) = 0 \end{aligned} \tag{A5.10}$$

Moreover, $M_{+-}(x,y)$ and $M_{-+}(x,y)$ differ multiplicatively only by a constant factor $-\rho_+/\rho_-$ (= $-b_+/b_-$). Observe that the interchange rule (9.9)-(9.12) holds here also.

When $k_\pm(y-x)$ is small, then to first order in $k_\pm(y-x)$ the quantities in (A5.6)-(A5.9) may be reduced, with the help of (A4.28) and (A4.29), to

$$M_{++}(x,y) = 1 - \tau_+(y-x) \tag{A5.11}$$

$$M_{-+}(x,y) = -\rho_-(y-x) \tag{A5.12}$$

$$M_{+-}(x,y) = \rho_+(y-x) \tag{A5.13}$$

$$M_{--}(x,y) = 1 + \tau_{--}(y-x) \tag{A5.14}$$

These relations may be compared to (9.3a)-(9.6a).

6. *Reflectances and Transmittances for a Homogeneous Layer*

The formulas (10.3)-(10.6) that convert the fundamental operators $\underline{M}_{++},\ldots,\underline{M}_{--}$ to reflectances and transmittances $\underline{R}(y,\underline{x}),\ldots,\underline{T}(y,x)$ of a water layer X[x,y] can be applied to the present scalar case also. Thus using (A5.6)-(A5.9) in those earlier formulas, we find, for $x \le y \le z$,

$$R(y,x) = -\rho_+\delta[e^{k_+(y-x)} - e^{k_-(y-x)}][\rho_+\rho_-\ e^{k_+(y-x)} - \delta^2\ e^{k_-(y-x)}]^{-1} \tag{A6.1}$$

$$T(x,y) = \Delta e^{(k_++k_-)(y-x)}\ [\rho_+\rho_-\ e^{k_+(y-x)} - \delta^2\ e^{k_-(y-x)}]^{-1} \tag{A6.2}$$

$$R(x,y) = -\rho_-\delta[e^{k_+(y-x)} - e^{k_-(y-x)}][\rho_+\rho_-\ e^{k_+(y-x)} - \delta^2\ e^{k_-(y-x)}]^{-1} \tag{A6.3}$$

$$T(y,x) = \Delta[\rho_+\rho_-\ e^{k_+(y-x)} - \delta^2\ e^{k_-(y-x)}]^{-1} \tag{A6.4}$$

From these relations we can read off various important physical properties of the optical medium. For example,

$$R(x,y) = (\rho_-/\rho_+)\ R(y,x) \quad [= (D_-/D_+)\ R(y,x)] \tag{A6.5}$$

and

$$T(x,y) = e^{(k_+ + k_-)(y-x)}\ T(y,x) = e^{(\tau_- - \tau_+)(y-x)}\ T(y,x) \tag{A6.6}$$

The parenthetical statement holds if we adopt the assumption $b_{\pm} = D_{\pm}\bar{b}$, as in §A2 above. Thus we see that upward and downward slab transfer properties are equal if and only if we have lighting isotropy ($D_+ = D_-$), as observed earlier. When $k_{\pm}(y-x)$ is small, (A6.1)-(A6.4) reduce, to first order in $k_{\pm}(y-x)$, to

$$R(y,x) = \rho_+(y-x) \tag{A6.7}$$

$$T(x,y) = 1 + \tau_-(y-x) \tag{A6.8}$$

$$R(x,y) = \rho_-(y-x) \tag{A6.9}$$

$$T(y,x) = 1 + \tau_+(y-x) \tag{A6.10}$$

When $k_{\pm}(y-x)$ is large, then

$$T(x,y) \simeq \frac{\Delta}{\rho_+\rho_-} \cdot e^{k_-(y-x)} \tag{A6.11}$$

$$T(y,x) \simeq \frac{\Delta}{\rho_+\rho_-} \cdot e^{-k_+(y-x)} \tag{A6.12}$$

In the limit of infinitely deep media, we find, from (A6.3), with the help of (A4.27), (A4.30), and (A4.31) that

$$R_-(\infty) \equiv \lim_{y\to\infty} R(x,y) = -\delta/\rho_+ = -(\tau_+ + k_+)/\rho_+ = (a_+ + b_+ - k_+)/b_+ \tag{A6.13}$$

$$= -\rho_-/(\tau_- - k_+) = b_-/(a_- + b_- + k_+) \tag{A6.14}$$

$$= -e_{-+}e_{++}^{-1} = f_{--}^{-1}\, f_{-+} \tag{A6.15}$$

The last equality, (A6.15) (which stems from (A4.7)), shows the formal connection with the radiance case, namely (11.3). The alternate formulas (involving $a_\pm$, $b_\pm$, $k_\pm$) are model versions of exact relations; see H.O., Vol. V, p. 113. Furthermore, from (A6.1), again with the help of (A4.26), (A4.29), and (A4.30):

$$R_+(\infty) \equiv \lim_{y\to\infty} R(y,x) = -\delta/\rho_- = -(\tau_- + k_-)/\rho_- = (a_- + b_- - k_-)/b_- \tag{A6.16}$$

$$= -\rho_+/(\tau_+ - k_-) = b_+/(a_+ + b_+ + k_-) \tag{A6.17}$$

$$= -e_{+-}e_{--}^{-1} = f_{++}^{-1}\, f_{+-} \tag{A6.18}$$

The last equality, (A6.18), as (A6.15), shows the formal connection with the isotropic radiance case. Observe how (11.12) and (11.14) reduce to the present formulas if one momentarily allows isotropy in the preceding equations and relaxes the non-commutativity property of matrix multiplication in section 11 (think of the matrices as 1×1).

7. *Solution for a Finitely Deep Medium*

We now may assemble the pieces of the solution of (A2.1) for a light field in a finitely deep medium such as that shown in Fig. 1. We assume given the set of four irradiance reflectances and transmittances $t(a,x)$, $t(x,a)$,

$r(a,x)$, $r(x,a)$ of the wind-blown surface $X[a,x]$, as generated in Preisendorfer and Mobley (1985, 1986). The optical properties of the homogeneous water body $X[x,z]$ are specified by the $D_{\pm}$, a, and $\bar{b}$ parameters in (A2.1). Finally, the reflectance $R(z,b)$ of the bottom $X[z,b]$ is assumed specified. Downward irradiance $H(a,-)$ is incident on the upper surface $X[a,x]$ and there are no other sources of flux on or in $X[a,b] = X[a,x] \cup X[x,z] \cup X[z,b]$. It is required to find $H(y,\pm)$ for all depths y, $x \le y \le z$, and also the emergent irradiance $H(a,+)$.

We start with the mapping (A3.5). Using (A5.2)-(A5.5) we find, on rearranging the terms:

$$\boxed{H(y,\pm) = a^{+}(x)\, e^{k_{+}(y-x)}\, f_{+\pm} + a^{-}(x)\, e^{k_{-}(y-x)}\, f_{-\pm}} \qquad \text{(A7.1)}$$

where

$$\begin{aligned} a^{+}(x) &\equiv H(x,+)\, e_{++} + H(x,-)\, e_{-+} \\ a^{-}(x) &\equiv H(x,+)\, e_{+-} + H(x,-)\, e_{--} \end{aligned} \qquad \text{(A7.2)}$$

Now consider the composite medium $X[x,b] = X[x,z] \cup X[z,b]$ consisting of the water body $X[x,z]$ and the reflecting lower boundary $X[z,b]$. Let us assign a reflectance $R(x,b)$ to $X[x,b]$, which we will later show how to evaluate. For the present, the global interaction principle applied to $X[x,b]$ yields the relation

$$H(x,+) = H(x,-)\, R(x,b) \qquad \text{(A7.3)}$$

The amplitudes $a^{\pm}(x)$ in (A7.1) and (A7.2) then reduce, with the help of (A6.15) and (A6.18), to

$$a^{+}(x) = H(x,-)\ [R(x,b) - R_{-}(\infty)]\ e_{++} \tag{A7.4}$$

$$a^{-}(x) = H(x,-)\ [1 - R(x,b)\ R_{+}(\infty)]\ e_{--} \tag{A7.5}$$

We therefore may evaluate $H(y,\pm)$ at every level y in X[x,z], provided we know H(x,-) and R(x,b) (cf. (16.2) and (16.3)).

Now, by the scalar version of the union rule in §15, for X[x,z] and X[z,b] we have

$$R(x,b) = R(x,z) + \mathcal{R}(x,z,b)\ T(z,x) \tag{A7.6}$$

where

$$\mathcal{R}(x,z,b) = T(x,z)\ [1 - R(z,b)\ R(z,x)]^{-1}\ R(z,b) \tag{A7.7}$$

The quartet R(z,x), T(x,z), R(x,z), T(z,x) is found via (A6.1)-(A6.4), while R(z,b) is given.

To find H(x,-), we use the scalar version of the imbed rule in §14 to first of all determine

$$\mathcal{T}(a,x,b) = t(a,x)[1 - R(x,b)\ r(x,a)]^{-1} \tag{A7.8}$$

where t(a,x) and r(x,a) are the air-water surface's downward irradiance transmittance and upward irradiance reflectance, respectively. Then

$$H(x,-) = H(a,-)\ \mathcal{T}(a,x,b) \tag{A7.9}$$

The emergent flux H(a,+) at level a is given by the global interaction principle applied to X[a,x]:

$$H(a,+) = H(a,-)\ r(a,x) + H(x,+)\ t(x,a) \tag{A7.10}$$

where H(x,+) is given by (A7.3), and r(a,x) and t(x,a) are the remaining two irradiance transfer coefficients for the air-water surface. This completes the solution.

8. *Solution for an Infinitely Deep Medium*

In (A7.4) and (A7.5) set $R(x,b) = R_-(\infty)$ (which is the limit, as $z \to \infty$, of R(x,b) in (A7.6)). Then for $x \le y < \infty$, (A7.1) reduces to

$$H(y,\pm) = H(x,-)[1 - R_-(\infty)R_+(\infty)]\ e_{--}e^{k_-(y-x)}f_{-\pm} \tag{A8.1}$$

or simply

$$\boxed{H(y,-) = H(x,-)\Delta e^{k_-(y-x)}} \tag{A8.2}$$

and

$$\boxed{H(y,+) = H(y,-)\ R_-(\infty)} \tag{A8.3}$$

where Δ is given in (A4.29) and $R_-(\infty)$ in (A6.13). H(x,-) is given in (A7.9) with $b = \infty$, and H(a,+) is given by (A7.10). Thus we find

$$H(x,-) = H(a,-)\ t(a,x)\ [1 - R_-(\infty)\ r(x,a)]^{-1} \tag{A8.4}$$

$$H(x,+) = H(x,-)\ R_-(\infty) \tag{A8.5}$$

and

$$H(a,+) = H(a,-)\,\{r(a,x) + t(a,x)\,[1 - R_{-}(\infty)\,r(x,a)]^{-1}\,R_{-}(\infty)\,t(x,a)\} \quad (A8.6)$$

$$= H(a,-)\,R(a,\infty)$$

Here $R(a,\infty)$ is the limit, as $b \to \infty$, of $R(a,b)$ where by the union rule (15.1) applied to the union $X[a,b]$ of $X[a,x]$ and $X[x,b]$, we have

$$R(a,b) = r(a,x) + t(a,x)\,[1 - R(x,b)\,r(x,a)]^{-1}\,R(x,b)\,t(x,a) \quad (A8.7)$$

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